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An Examination of Equilibria in the
Multi-Site Iterated Prisoner's Dilemma

by

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A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of Master of Science in
Applied Mathematics

School of Mathematical Sciences
College of Science

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Abstract

Iterated Prisoner's Dilemma is a simple model for the interaction between two self-interested agents who can choose whether or not to cooperate with one another. Many real-world problems can be characterized in terms of the Iterated Prisoner's Dilemma, from the militarization of rival nations to the tradeoff between gas mileage and safety when purchasing a vehicle. The game theoretic properties of Iterated Prisoner's Dilemma are well understood, and previous research by Robert Axelrod into the performance of various strategies in a Darwinian environment is extensive. In this paper we extend Axelrod's work by investigating the dynamics of the Iterated Prisoner's Dilemma when multiple "communities," each playing its own Iterated Prisoner's Dilemma tournament, are allowed to interact and influence each other. Specifically, we examine the case when these communities are populated by players using two specific strategies: TIT-FOR-TAT and ALWAYS-NONCOOPERATIVE. We begin with rigorous analysis of the underlying structure of the model in order to determine the conditions under which one of the two player types faces extinction. We then demonstrate that cooperative equilibria do exist, but a linear stability analysis indicates that these equilibria are unreachable in our proposed system.

1 Introduction

The mathematical game of Prisoner's Dilemma is often used as a simple model of scenarios in which two independent, self-interested agents must choose whether or not to cooperate with each other. While mutual cooperation is more beneficial than mutual noncooperation, asymmetric decisions yield an even greater benefit to one agent (and a correspondingly poor result for the other). As its name implies, the logical structure of the game is illustrated by the story of a prisoner. The prisoner and his partner in crime are arrested and immediately separated for individual questioning. Although the authorities believe that the prisoners are guilty of a serious offense, there is only sufficient evidence to convict them of a lesser crime. For this reason, the authorities offer Prisoner A a deal: if he confesses to the serious offense and provides evidence that leads to the conviction of Prisoner B, he will receive a sentence of only probation, but Prisoner B will face significant jail time. Naturally, Prisoner B receives the same offer. In the case that *both* prisoners confess, the offer is considered void, but because their confessions have saved the authorities the time and money needed for prosecution, both prisoners will receive a reduced sentence. Implicit, but unspoken to the prisoners is the fact that if neither accepts the offer, they will both be convicted of the lesser crime and will face time behind bars, but not as much as if they had been convicted of the more serious offense.

If Prisoner A chooses to confess, then the self-interested Prisoner B is better off confessing as well in order to reduce his own sentence. If Prisoner A remains silent, it is still in the best interest of Prisoner B to confess to ensure that he receives no jail time. Prisoner A sees the same logic, and so both are led to confess even though they would have been better off if they had both chosen to remain silent.

Scenarios with a similar logical structure exist in a wide variety of contexts. For instance, Snyder [6] analyzes elements of international relations in light of the Prisoner's Dilemma, leading to the formalization of the "security dilemma," a phenomenon seen

in militarization, disarmament, alliances, and colonization. We consider the specific case of militarization between rival nations that are at peace but wary of one another. Either by treaty or unspoken cooperation, each maintains a predictable level of arms. If one of these countries chooses to be “noncooperative” by fortifying its military, and can do so quickly enough to develop first-strike capability, it can dictate terms to the other. This leads both countries to mutual noncooperation, and so both move to build up arms, which results in a significant loss of capital but not better security.

Consumers face a similar kind of “arms race” when deciding what kind of vehicle to purchase. As White [7] explains, families see larger vehicles as providing more protection in collisions, especially in cases where one vehicle (e.g., an SUV) has a higher profile than another (e.g., a compact car). If all families purchase small vehicles, all would be equally safe. If all families purchase larger vehicles instead, all are still equally safe, but they incur the added cost of reduced fuel efficiency. However, if most families purchase small vehicles, a single family can act in a “noncooperative” fashion and choose to buy a large vehicle, and that family is very safe while others are less so due to their “cooperative” choice.

The logical structure of such scenarios is encapsulated in the mathematical game of Prisoner’s Dilemma. In this game, players earn points based on both their decision and that of their opponents. As previously seen, the context of the game will dictate what choices each player can make, but in general we abstract the decision to either “cooperation” or “noncooperation”. We assume that the structure of the game is known to both players, that the players do not communicate except to reveal their decisions, and that the players are concerned only with maximizing their own score, or *payoff*. The payoff parameters for a particular game are often gathered in a *payoff matrix* such as

Choice of Player A	→ C	6, 6	3, 7
	→ N	7, 3	4, 4
		C	N
Choice of Player B		↑	↑

in which C indicates the choice of cooperation, N indicates the choice of noncooperation, and the ordered pair in each cell has the form (Payoff to Player A, Payoff to Player B). Given this particular matrix, both players receive a payoff of 6 if both choose to cooperate, but Player A receives 7 points if he chooses noncooperation when Player B chooses to cooperate. More generally, the payoff matrix for Prisoner's Dilemma has the form

Choice of Player A	→ C	R, R	S, T
	→ N	T, S	P, P
		C	N
Choice of Player B		↑	↑

where $T > R > P > S$. This string of inequalities is what defines the structure of Prisoner's Dilemma. Relaxing the inequalities or switching values effectively leads to the creation of a different game with an inherently different structure. The values are labeled according to their significance: R is the *reward* for mutual cooperation, P is the *penalty* for mutual noncooperation, T is the *temptation* to choose noncooperation, and S is the *sucker's payoff*, where the "sucker" is taken to be the player who is "tricked" into cooperating so that the opponent can reap significant benefit. These values are often taken to be positive, but the logical structure of the game only depends on the string of inequalities above. For the sake of discussion, suppose that Player B chooses to cooperate. Then Player A has a payoff of R if he cooperates, and a payoff of T otherwise. Since $T > R$, the self-interested Player A should choose noncooperation. Similarly, if Player B chooses noncooperation, the fact that $P > S$ leads Player A to do the same. In the language of mathematical game theory, noncooperation is said to be a

dominant strategy for Player A, meaning that Player A earns more points by choosing noncooperation regardless of the strategy employed by Player B. The same is true for Player B. Were both players to cooperate, each would earn the next-to-best number of points, but the dominant strategy of noncooperation leads self-interested players to make choices that result in earning the next-to-worst number of points instead.

The possibility of multiple interactions between the same players changes the situation. In this case, commonly called the Iterated Prisoner’s Dilemma (IPD), a game between two players can consist of more than one turn, and players can keep a running record of the choices made in previous turns of a game and change their strategy accordingly. For the purposes of this paper, we will consider an IPD tournament that hosts a large number of players and consists of three nested levels of interaction: a turn is a single choice made by each player in the context of the Prisoner’s Dilemma, a game consists of multiple turns one after the other, and a round is made up of all the games played amongst all players in the population. The tournament can consist of multiple rounds in which unsuccessful strategies are “eliminated” after each iteration. Loosely interpreted in a sociobiological sense, each round of the IPD tournament can be understood as a generation, each game between players as a relationship, and each turn of a game as a single interaction.

In the early 1980s, Axelrod reported experiments in which computer algorithms implemented the decision strategies submitted by experts in economics, psychology, sociology, political science, and mathematics (see [1], [2], and [3]). In the first round of this tournament, each of the 14 algorithms played a game of IPD against every other algorithm. At the conclusion of the round, algorithms that performed well according to certain measures were duplicated while those that did poorly were eliminated from the tournament. The procedure was repeated in subsequent rounds, and Axelrod kept track of the population demographics. One particular strategy, developed by Anatol Rapoport and called TIT-FOR-TAT, emerged as being particularly successful. The rules governing this strategy are surprisingly simple: (1) begin each game of IPD by coop-

erating, and (2) in each subsequent turn of the game, do what the opponent did in the previous turn. Thus when two TIT-FOR-TAT players meet, they begin by cooperating and do so for all rounds thereafter.

A simple mathematical analysis of the structure of IPD sheds light on why the TIT-FOR-TAT strategy is so effective. Let us suppose that the number of turns in a game of IPD is random, and that w is the probability that the next turn will take place (we take w to be a parameter of the tournament, not a specific game). Because future turns of a game might not happen, points earned in the current round have more value than potential points from future turns. This leads us to discount future payoffs by the probability that they will occur; the cumulative total of these weighted payoffs is the *present value* of the game, denoted by $V(\mathcal{S}_1, \mathcal{S}_2)$ where \mathcal{S}_1 is the player's strategy and \mathcal{S}_2 is the opponent's strategy. For example, the present value of a game of IPD between two TIT-FOR-TAT players is

$$V(\text{TFT}, \text{TFT}) = R + wR + w^2R + w^3R + \dots = \frac{R}{1-w},$$

since both players begin by cooperating and thus cooperate for every round thereafter. Similarly, the present value of a game between two players who always choose noncooperation, which we denote by MEAN, is

$$V(\text{MEAN}, \text{MEAN}) = P + wP + w^2P + w^3P + \dots = \frac{P}{1-w},$$

since both MEAN players will only ever choose noncooperation. When TIT-FOR-TAT and MEAN players meet, the present value of the game to the TIT-FOR-TAT player is

$$V(\text{TFT}, \text{MEAN}) = S + wP + w^2P + w^3P = S + \frac{wP}{1-w},$$

and the present value to the MEAN player is

$$V(\text{MEAN}, \text{TFT}) = T + wP + w^2P + w^3P = T + \frac{wP}{1-w}.$$

These two present values differ only by the score in the first round since a TIT-FOR-TAT player will mimic the MEAN player after the initial loss. For the sake of discussion,

let us suppose that the population of algorithms are of only these two types, and that $x \in [0, 1]$ is the fraction of the population that plays TIT-FOR-TAT. Then a TIT-FOR-TAT player in a round-robin tournament should expect the value of a game to be, on average,

$$\mathcal{E}_T(x) = xV(\text{TFT}, \text{TFT}) + (1 - x)V(\text{TFT}, \text{MEAN}), \quad (1)$$

and MEAN players should expect to score, on average,

$$\mathcal{E}_M(x) = xV(\text{MEAN}, \text{TFT}) + (1 - x)V(\text{MEAN}, \text{MEAN}) \quad (2)$$

points per game. Axelrod's tournament was designed to encourage a Darwinian process of selection in order to determine the best strategy: successful strategies experienced significant growth while unsuccessful strategies became extinct. In an evolutionary setting such as Axelrod's tournament, we expect the population of TIT-FOR-TAT players to grow relative to the population of MEAN players when $\mathcal{E}_T(x) > \mathcal{E}_M(x)$, vice versa when $\mathcal{E}_M(x) > \mathcal{E}_T(x)$, and we expect demographic equilibrium when $\mathcal{E}_T(x) = \mathcal{E}_M(x)$. By substituting the appropriate values of $V(\mathcal{S}_1, \mathcal{S}_2)$ into equations (1) and (2), we see that $\mathcal{E}_T(x) = \mathcal{E}_M(x)$ if and only if

$$\begin{aligned} xV(\text{TFT}, \text{TFT}) + (1 - x)V(\text{TFT}, \text{MEAN}) &= xV(\text{MEAN}, \text{TFT}) + (1 - x)V(\text{MEAN}, \text{MEAN}) \\ x \left(\frac{R}{1 - w} \right) + (1 - x) \left(S + \frac{wP}{1 - w} \right) &= x \left(T + \frac{wP}{1 - w} \right) + (1 - x) \left(\frac{P}{1 - w} \right). \end{aligned}$$

Multiplying both sides by $(1 - w)$ and collecting all terms containing x on the left-hand side, we have that

$$[P - S + R - T + (S + T - 2P)w]x = (P - S)(1 - w). \quad (3)$$

This relationship between x and w is depicted as a curve in the xw -plane in Figure 1. The point Q in the figure represents a situation in which TIT-FOR-TAT players are more numerous than needed to balance the first-turn losses to players that employ the MEAN strategy, so $\mathcal{E}_T(x) > \mathcal{E}_M(x)$. Consequently, the TIT-FOR-TAT population should experience larger growth than the MEAN population, so we expect x to increase from one round of the tournament to the next and the point Q will move right along a horizontal

line. Similarly, the point Y represents a scenario in which there are fewer TIT-FOR-TAT players than needed for equilibrium. In this case, we expect the demographics to shift in favor of the MEAN players from one round to the next, so the point Y will move left along a horizontal line until the TIT-FOR-TAT player type is eliminated from the tournament. In the language of evolutionary game theory, the TIT-FOR-TAT “species” is driven to extinction over time if the population fraction x is too small relative to the parameter w . Let us also note that Figure 1 shows no x at which $\mathcal{E}_T(x) = \mathcal{E}_M(x)$ when w is small. This happens because the average length of a game, $1/(1-w)$, must be sufficiently large in order for the benefits of mutual cooperation between TIT-FOR-TAT players to overcome the deficit they incur in the first turn of each game against MEAN players.

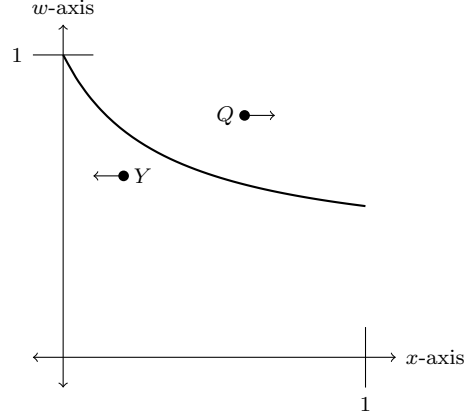


Figure 1: The graph of equation (3) when $T = 11, R = 7, P = 3, S = 1$ where w is the probability of continuing play in a game of IPD and x is the TIT-FOR-TAT population in the tournament. Y is the case when the TIT-FOR-TAT population goes extinct and Q is the case when TIT-FOR-TAT players eventually dominate the tournament.

In the work that follows, we will consider an IPD tournament in which $T = 7, R = 6, P = 4$, and $S = 3$. With these parameters, we can use equation (3) to determine the fraction of TIT-FOR-TAT players required for evolutionary stability in the single site,

henceforth denoted by λ , as

$$\begin{aligned}
[P - S + R - T + (S + T - 2P)w] \lambda &= (P - S)(1 - w) \\
[4 - 3 + 6 - 7 + (3 + 7 - 2(4))w] \lambda &= (4 - 3)(1 - w) \\
2w\lambda &= 1 - w \\
\lambda &= \frac{1 - w}{2w}.
\end{aligned} \tag{4}$$

Thus by definition, $\mathcal{E}_T(\lambda) = \mathcal{E}_M(\lambda)$. It will often be more convenient to treat λ as a parameter for the system rather than w , and so we invert this function to determine w in terms of λ as

$$w = \frac{1}{1 + 2\lambda} \tag{5}$$

Note that $\lambda = 0.5$ when $w = 0.5$, and $0 < \lambda < 1$ only when $w > 1/3$. This says that there exists an equilibrium only if the games are sufficiently long, which is exactly the conclusion taken from Figure 1.

Axelrod extended his simulations to scenarios in which a population is not well-mixed in order to study the *territorial stability* of strategies. In these scenarios, individuals are distributed across a lattice, and interact only with neighbors. Nakamaru et al. [4] found that TIT-FOR-TAT players form tight clusters in the one-dimensional lattice and that these clusters can spread when w is sufficiently large. In related work, Oliphant [5] demonstrated that the spatial organization of player types on a spatial lattice can have a significant effect on the evolution of a population.

In this paper we consider a modification of the lattice model in which vertices on the lattice represent distinct but interacting communities. Each community is well-mixed, the interaction between individuals is modeled with IPD, and the state of each node is a continuous variable in $[0, 1]$ that indicates the fraction of its population that plays the TIT-FOR-TAT strategy. More specifically, we examine three such communities, or *sites*, that are situated along a “road”, and in each round of the tournament some fraction of the population from each site can travel to tournaments in neighboring sites.

The remainder of this thesis is organized as follows. In section 2 we construct a replicator dynamic for population growth in the model and establish initial conditions for the multi-site tournament. We then use simulations to show that given these initial conditions the population of initially neutral sites can be driven fully to either player type in the long-term limit. In sections 3 and 4 we establish the underlying mathematical structure of the game and prove a number of facts regarding the governing equations of the system. In sections 5 and 6 we prove that one player type or the other is driven to extinction given certain conditions on the parameters of the IPD tournament. In section 7 we analyze the existence and stability of nontrivial equilibria in which the demographics of the central population stabilizes, but neither player type is driven to extinction. Finally, in section 8 we use numerical simulations in order to estimate the value of the nontrivial equilibrium and its dependence on the fraction of players who travel from their home site.

2 Mathematical Representation

In this section we develop a mathematical model of an IPD tournament with a focus on the growth of opposing populations over time. This evolutionary mechanism will update both the TIT-FOR-TAT and MEAN populations based on their relative success as measured by the expected value of a round. We begin by constructing a replicator dynamic for the single-site tournament and then extend this idea to the multi-site tournament. We conclude with simulations of the multi-site model that demonstrate how either player type can dominate initially neutral sites in the long-term limit. As seen in Section 1, we will assume that the payoff matrix is fixed with $T = 7$, $R = 6$, $P = 4$, and $S = 3$.

2.1 The Single-Site Model

In the case of a single site, we denote by x_n the fraction of the population in round n that plays the TIT-FOR-TAT strategy, and by y_n the complementary fraction of MEAN players. Since there are only these two types of players in the tournament,

$$1 = x_n + y_n \quad \text{for all } n. \tag{6}$$

The evolutionary mechanism in our model is based on the following principle: if one population does twice as well as the other, it experiences twice the relative growth as the other. This is expressed by the equation

$$\frac{x_{n+1}}{x_n} = \left(\frac{\mathcal{E}_T(x_n)}{\mathcal{E}_M(x_n)} \right) \frac{y_{n+1}}{y_n}. \tag{7}$$

We now combine equations (6) and (7) using substitution to arrive at a direct formulation for x_{n+1} .

$$\begin{aligned}
\frac{x_{n+1}}{x_n} &= \left(\frac{\mathcal{E}_T}{\mathcal{E}_M} \right) \frac{1 - x_{n+1}}{y_n} \\
(x_{n+1})(y_n)\mathcal{E}_M &= (1 - x_{n+1})(x_n)\mathcal{E}_T \\
x_{n+1}(y_n\mathcal{E}_M + x_n\mathcal{E}_T) &= x_n\mathcal{E}_T \\
x_{n+1} &= \frac{x_n\mathcal{E}_T}{x_n\mathcal{E}_T + y_n\mathcal{E}_M}.
\end{aligned} \tag{8}$$

The complimentary fraction of MEAN players is

$$y_{n+1} = 1 - x_{n+1} = \frac{y_n\mathcal{E}_M}{x_n\mathcal{E}_T + y_n\mathcal{E}_M}.$$

Equation (8) allows us to formulate the increment in the TIT-FOR-TAT population from round n to round $n + 1$, which we will denote by $\Delta(x_n)$, as

$$\begin{aligned}
\Delta(x_n) &= x_{n+1} - x_n \\
&= \frac{x_n\mathcal{E}_T}{x_n\mathcal{E}_T + (1 - x_n)\mathcal{E}_M} - x_n.
\end{aligned}$$

We can substitute the expressions for $\mathcal{E}_T(x_n)$ and $\mathcal{E}_M(x_n)$ into the increment as defined above to obtain an expression in x_n . Based on equation (1) we know

$$\begin{aligned}
\mathcal{E}_T(x_n) &= x_n \left(\frac{6}{1 - w} \right) + (1 - x_n) \left(3 + \frac{4w}{1 - w} \right) \\
&= \left(\frac{1}{1 - w} \right) (6x_n + (1 - x_n)(3(1 - w) + 4w)) \\
&= \left(\frac{1}{1 - w} \right) (3x_n + 3 + w - wx_n).
\end{aligned} \tag{9}$$

Similarly, equation (2) tells us

$$\begin{aligned}
\mathcal{E}_M(x_n) &= x_n \left(7 + \frac{4w}{1 - w} \right) + (1 - x_n) \left(\frac{4}{1 - w} \right) \\
&= \left(\frac{1}{1 - w} \right) (x_n(7(1 - w) + 4w) + (1 - x_n)(4)) \\
&= \left(\frac{1}{1 - w} \right) (3x_n - 3wx_n + 4).
\end{aligned} \tag{10}$$

We can now substitute these results into the increment $\Delta(x_n)$. Note that the factor of $\frac{1}{1-w}$ in the formulas for both \mathcal{E}_T and \mathcal{E}_M will cancel once substituted into $\Delta(x)$. It is also helpful to note that the increment does not depend on the round number, so we can replace x_n with x for simplicity. We write the increment as

$$\begin{aligned}\Delta(x) &= \frac{x(3x + 3 + w - wx)}{x(3x + 3 + w - wx) + (1-x)(3x - 3wx + 4)} - x \\ &= \frac{(x - x^2)(3x + 3 + w - wx) - x(1-x)(3x - 3wx + 4)}{x(3x + 3 + w - wx) + (1-x)(3x - 3wx + 4)}.\end{aligned}$$

Factoring the numerator and simplifying the denominator, we have

$$\Delta(x) = \frac{x(1-x)(2wx + w - 1)}{2w(x-1)x + 2x + 4}.$$

We use the identity between w and λ seen in equation (4) to simplify the numerator even further:

$$\begin{aligned}\Delta(x) &= \frac{2wx(1-x) \left(x - \frac{1-w}{2w}\right)}{2wx(x-1) + 2x + 4} \\ &= \frac{2wx(1-x)(x - \lambda)}{2wx(x-1) + 2x + 4}.\end{aligned}\tag{11}$$

Note that this function is zero at $x = 0$, $x = 1$, and $x = \lambda$. Further, since $w \in [0, 1]$,

$$\begin{aligned}2w(x-1)x + 2x + 4 &= 2x(w(x-1) + 1) + 4 \\ &= 2x((1-w) + wx) + 4 > 0.\end{aligned}$$

hence the polarity of the increment is determined solely by the factor $(x - \lambda)$ in the numerator. A typical graph of $\Delta(x)$ is show in Figure 2, which demonstrates that $\Delta(x_n) > 0$ only when $x_n \in (\lambda, 1)$. In the lemma below we prove that the sequence $\{x_n\}$ is increasing and bounded above and that $x_n \rightarrow 1$ as $n \rightarrow \infty$ provided $x_1 \in (\lambda, 1]$.

Lemma 1. *Suppose $x_1 \in (\lambda, 1]$ in the single-site scenario, and $x_{n+1} = x_n + \Delta(x_n)$ as stated previously. Then $x_n \rightarrow 1$ as $n \rightarrow \infty$. Similarly, $x_n \rightarrow 0$ as $n \rightarrow \infty$ if $x_1 \in [0, \lambda]$ in the single-site scenario.*

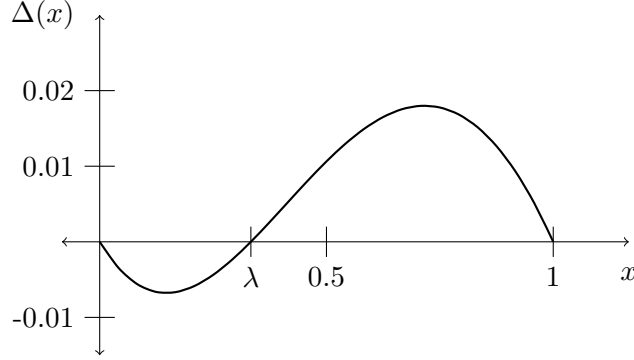


Figure 2: A typical graph of $\Delta(x)$, here with $\lambda = 1/3$.

Proof. First, suppose that $x_1 = 1$. We know that $\Delta(1) = 0$, hence $x_{n+1} = x_n = 1$ for all $n \geq 1$.

Now let $x_1 \in (\lambda, 1)$. We can determine the value of x_n for subsequent rounds using equation (8), and since the denominator in equation (11) is clearly greater than the numerator, it follows that $x_n \leq 1$ for all $n \in \mathbb{N}$. Since $x_1 \in (\lambda, 1)$, we have that $x_1 - \lambda > 0$, and so equation (11) tells us that $\Delta(x_1) > 0$. Since $x_2 > \lambda$, we can reapply this argument, and so for all $n \in \mathbb{N}$, $x_{n+1} > x_n$. Hence the sequence $\{x_n\}$ is increasing and bounded above by 1, so it must converge to some value L in $(\lambda, 1]$. Using the properties of limits,

$$\begin{aligned}
0 &= L - L \\
&= \lim_{n \rightarrow \infty} x_{n+1} - \lim_{n \rightarrow \infty} x_n \\
&= \lim_{n \rightarrow \infty} (x_{n+1} - x_n) \\
&= \lim_{n \rightarrow \infty} \Delta(x_n).
\end{aligned}$$

Since $\Delta(x)$ is a continuous function on the interval $[0, 1]$,

$$0 = \Delta\left(\lim_{n \rightarrow \infty} x_n\right) = \Delta(L).$$

The only root of the function $\Delta(x)$ on the interval $(\lambda, 1]$ is $x = 1$, hence $L = 1$.

We now wish to consider the case when $x_1 \in [0, \lambda)$. Again, if $x_1 = 0$, we have that $\Delta(0) = 0$, and so $x_{n+1} = x_n = 0$ for all $n \in \mathbb{N}$.

Suppose $x_1 \in (0, \lambda)$. We again use equation (8) to establish that $x_{n+1} > 0$ and equation (11) to determine that $\Delta(x_n) < 0$ for all $n \in \mathbb{N}$, so $x_{n+1} < x_n$ for all rounds. Hence the sequence $\{x_n\}$ is decreasing and bounded below, so it must converge. Further, as seen above,

$$0 = L - L = \Delta\left(\lim_{n \rightarrow \infty} x_n\right) = \Delta(L).$$

The only root of $\Delta(x)$ on the interval $[0, \lambda)$ is $x = 0$, therefore $L = 0$ in this case. \square

2.2 The Three-Site Model

Now we adapt the single-site model to the case of three sites of equal population, say A, B, and C, that are distributed along a “road.” The fractions of TIT-FOR-TAT players at these sites in round n of the tournament are a_n , b_n , and c_n , respectively. Although each site holds its own IPD tournament, we assume that some fraction of the players from each site, say $q \in (0, 1)$, also interact with players from neighboring sites at intermediate “trading posts” (see Figure 3). Further, we assume that neither player type is more likely to travel than the other, so that q is both the fraction of TIT-FOR-TAT players and the fraction of MEAN players that travel to the trading post.

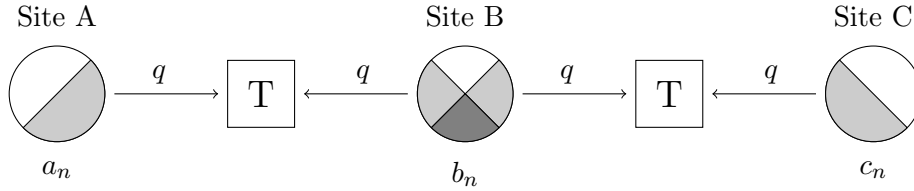


Figure 3: Schematic for player travel in the three-site line, where q is $1/2$.

In order to demonstrate the effect of the trading posts, let us partition the residents of Site A into two cohorts: those who travel to the trading post, and those who do not. Individuals in the latter cohort play games of IPD only with other residents of Site A,

so the expected value of a game to players in this cohort is either $\mathcal{E}_T(a_n)$ or $\mathcal{E}_M(a_n)$, according to which strategy they employ. Individuals in the other cohort encounter both the players from their own site and some number from Site B, thus the adjusted fraction of TIT-FOR-TAT players is

$$\frac{a_n + qb_n}{1 + q}.$$

So the expected value of a game for the traveling TIT-FOR-TAT players from Site A is

$$\mathcal{E}_T\left(\frac{a_n + qb_n}{1 + q}\right) = \left(\frac{a_n + qb_n}{1 + q}\right) V(\text{TFT}, \text{TFT}) + \left(1 - \frac{a_n + qb_n}{1 + q}\right) V(\text{TFT}, \text{MEAN}).$$

In order to quantify the overall “success” of TIT-FOR-TAT players from Site A, we combine the two cohorts with a weighted average according to the fraction of the TIT-FOR-TAT population in each, and so

$$\mathcal{E}_T^A = q\mathcal{E}_T\left(\frac{a_n + qb_n}{1 + q}\right) + (1 - q)\mathcal{E}_T(a_n). \quad (12)$$

Recall from equation (1) that

$$\begin{aligned} \mathcal{E}_T(x) &= xV(\text{TFT}, \text{TFT}) + (1 - x)V(\text{TFT}, \text{MEAN}) \\ &= (V(\text{TFT}, \text{TFT}) - V(\text{TFT}, \text{MEAN}))x + V(\text{TFT}, \text{MEAN}), \end{aligned}$$

which we write as

$$\mathcal{E}_T = mx + b$$

with $m = V(\text{TFT}, \text{TFT}) - V(\text{TFT}, \text{MEAN})$ and $b = V(\text{TFT}, \text{MEAN})$. Thus for any values x and y ,

$$\begin{aligned} \mathcal{E}_T^A &= q\mathcal{E}_T(x) + (1 - q)\mathcal{E}_T(y) \\ &= q(mx + b) + (1 - q)(my + b) \\ &= m(qx + (1 - q)y) + b \\ &= \mathcal{E}_T(qx + (1 - q)y). \end{aligned}$$

Based on equation (12), we let

$$x = \frac{a_n + qb_n}{1 + q} \quad \text{and} \quad y = a_n,$$

and so it follows that,

$$\mathcal{E}_T^A = \mathcal{E}_T \left(q \left(\frac{a_n + qb_n}{1 + q} \right) + (1 - q)a_n \right).$$

This leads us to define the *effective* TIT-FOR-TAT population fraction for Site A as

$$p_A = q \left(\frac{a_n + qb_n}{1 + q} \right) + (1 - q)a_n.$$

Similarly, we can derive such a formulation for the effective population in Site C as

$$p_C = q \left(\frac{c_n + qb_n}{1 + q} \right) + (1 - q)c_n.$$

Whereas the players in Sites A and C have only one direction of travel, players in Site B have two, hence there are four possible behaviors: no travel, travel to one site (either A or C), or travel to both sites. The table below lists the probability of each behavior and the associated fractions of TIT-FOR-TAT players encountered.

Behavior	Probability	TIT-FOR-TAT Fraction
Travel to neither A nor C	$(1 - q)^2$	b_n
Travel only to Site A	$q(1 - q)$	$\frac{b_n + qa_n}{1 + q}$
Travel only to Site C	$q(1 - q)$	$\frac{b_n + qc_n}{1 + q}$
Travel to both A and C	q^2	$\frac{qa_n + b_n + qc_n}{1 + 2q}$

As before, we determine the overall success for players in Site B by considering a weighted average of the expected scores for each cohort of players.

$$\begin{aligned} \mathcal{E}_T^B = & (1 - q)^2 \mathcal{E}_T(b_n) + q(1 - q) \mathcal{E}_T \left(\frac{qa_n + b_n}{1 + q} \right) \\ & + q(1 - q) \mathcal{E}_T \left(\frac{b_n + qc_n}{1 + q} \right) + q^2 \mathcal{E}_T \left(\frac{qa_n + b_n + qc_n}{1 + 2q} \right) \end{aligned}$$

Using a similar strategy as seen with the derivation for \mathcal{E}_T^A , we consider a general summation of the form

$$\mathcal{E}_T^* = \sum_{i=0}^N q_i \mathcal{E}_T(x_i) \quad \text{with} \quad \sum_{i=0}^N q_i = 1,$$

where N is a positive integer and all x_i are positive real numbers. Since $\mathcal{E}_T(x)$ has the form $\mathcal{E}_T(x) = mx + b$,

$$\begin{aligned} \mathcal{E}_T^* &= \sum_{i=0}^N q_i (mx_i + b) \\ &= m \sum_{i=0}^N q_i x_i + b \sum_{i=0}^N q_i \\ &= m \left(\sum_{i=0}^N q_i x_i \right) + b \\ &= \mathcal{E}_T \left(\sum_{i=0}^N q_i x_i \right). \end{aligned}$$

It is clear to see that

$$(1 - q)^2 + 2q(1 - q) + q^2 = 1,$$

so the expected score in Site B can be written as

$$\mathcal{E}_T^B = \mathcal{E}_T \left((1 - q)^2 b_n + q(1 - q) \left(\frac{qa_n + 2b_n + qc_n}{1 + q} \right) + q^2 \left(\frac{qa_n + b + qc_n}{1 + 2q} \right) \right).$$

Hence we define the effective population for Site B to be

$$p_B = (1 - q)^2 b_n + q(1 - q) \left(\frac{qa_n + 2b_n + qc_n}{1 + q} \right) + q^2 \left(\frac{qa_n + b + qc_n}{1 + 2q} \right).$$

When we rewrite the effective populations by collecting coefficients for a_n , b_n , and c_n , we see that

$$\begin{aligned} p_A &= q \left(\frac{a_n + qb_n}{1 + q} \right) + (1 - q)(a_n) \\ &= \left(\frac{q}{1 + q} + (1 - q) \right) a_n + \left(\frac{q^2}{1 + q} \right) b_n \\ &= \left(\frac{1 + q - q^2}{1 + q} \right) a_n + \left(\frac{q^2}{1 + q} \right) b_n. \end{aligned} \tag{13}$$

Since Sites A and C are symmetric in their formulation for effective populations,

$$p_C = \left(\frac{1+q-q^2}{1+q} \right) c_n + \left(\frac{q^2}{1+q} \right) b_n. \quad (14)$$

A similar derivation for p_B allows us to rewrite the effective population as

$$p_B = \left(\frac{q^2+2q^3-q^4}{1+3q+2q^2} \right) a_n + \left(\frac{1+3q-4q^3+2q^4}{1+3q+2q^2} \right) b_n + \left(\frac{q^2+2q^3-q^4}{1+3q+2q^2} \right) c_n. \quad (15)$$

Since $q \in (0, 1)$, these coefficients are all positive and sum to 1. These forms for the expected values p_A , p_B , and p_C are useful as they demonstrate that the effective populations are simply weighted averages of the three TIT-FOR-TAT population fractions: a_n , b_n , and c_n .

Here we notice that this population mixing will change the pool of opponents that a particular player expects to meet compared to the single-site scenario, and this in turn affects the expected value of a game to that player. For this reason, we reconsider equations (1) and (2) for Sites A, B, and C, and take $x = p_A$, p_B , or p_C respectively. The resulting values of \mathcal{E}_T and \mathcal{E}_M are then used to update each site's TIT-FOR-TAT population using equations (6) and (7); however, because the players do not migrate from one site to another, the resident population, denoted by x_n in equations (6) and (7), is instead replaced with a_n , b_n , or c_n as appropriate. We can use these equations to construct a multi-site analog of the single-site increment which we will denote by I_j where $j \in \{A, B, C\}$. We will demonstrate this process for Site A, but the derivation is equivalent for the other two sites.

As with the single-site increment, the increment for Site A is defined as

$$\begin{aligned} I_A &= a_{n+1} - a_n \\ &= \frac{a_n \mathcal{E}_T^A}{a_n \mathcal{E}_T^A + (1 - a_n) \mathcal{E}_M^A} - a_n. \end{aligned}$$

The values of \mathcal{E}_T^A and \mathcal{E}_M^A are determined by equations (9) and (10) when $x = p_A$, thus

$$\begin{aligned} \mathcal{E}_T^A &= p_A V(\text{TFT}, \text{TFT}) + (1 - p_A) V(\text{TFT}, \text{MEAN}) \\ &= \left(\frac{1}{1-w} \right) (3p_A + 3 + w - wp_A) \end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_M^A &= p_A V(\text{MEAN}, \text{TFT}) + (1 - p_A) V(\text{MEAN}, \text{MEAN}) \\ &= \left(\frac{1}{1 - w} \right) (3p_A - 3wp_A + 4).\end{aligned}$$

We now substitute these results into the formulation for the increment. As in our previous work, we note that the increment does not depend on the round number, so we will replace a_n with just a :

$$\begin{aligned}I_A &= \frac{a(3p_A + 3 + w - wp_A)}{a(3p_A + 3 + w - wp_A) + (1 - a)(3p_A - 3wp_A + 4)} - a \\ &= \frac{a(1 - a)(2wp_A + w - 1)}{2wp_A a + 4 + (1 - w)(3p_A - a)}.\end{aligned}$$

Using the identity between w and λ from equation (4), we rewrite this as

$$\begin{aligned}I_A &= \frac{2wx(1 - x)(p_A - \lambda)}{2w(p_A x + \frac{2}{w} + \lambda(3p_A - x))} \\ &= \frac{x(1 - x)(p_A - \lambda)}{p_A x + 2 + 4\lambda + \lambda(3p_A - x)}.\end{aligned}$$

The increments for Sites B and C have the same form and can be obtained by replacing a and p_A appropriately. We will often consider the increment for a general site j , with $j \in \{A, B, C\}$, as

$$I_j(x) = \frac{x(1 - x)(p - \lambda)}{px + 2 + 4\lambda + \lambda(3p - x)}. \quad (16)$$

The main difference between the single-site and multi-site increment is that there is now a root at $p = \lambda$ rather than $x = \lambda$, and the polarity depends on the factor of $(p - \lambda)$ in the numerator.

2.3 Three-Site Simulations

Simulations allow us to observe the effect of Sites A and C on the demographics of the central community, Site B. For several values of the single-site equilibrium, λ , we initialize Site B with a TIT-FOR-TAT population of exactly λ . This population of TIT-FOR-TAT players is said to have a *growth potential* of $1 - \lambda$, and the initial population

in Site A is biased in favor of TIT-FOR-TAT players by some fraction of the growth potential. The population in Site C is biased in favor of the MEAN strategy by the same fraction of the MEAN growth potential, which is λ . Thus we begin with

$$\begin{aligned} a_1 &= \lambda + \varepsilon(1 - \lambda) \\ b_1 &= \lambda \\ c_1 &= \lambda - \varepsilon\lambda \end{aligned} \tag{17}$$

where ε is a small positive constant. Figure 4 shows results for $\lambda \in \{0.5, 0.43\}$, $\varepsilon = 0.1$, and $q = 0.5$. These simulations indicate that the single-site equilibrium, λ , is not necessarily an equilibrium in the multi-site scenario, and either player type can dominate Site B in the long run.

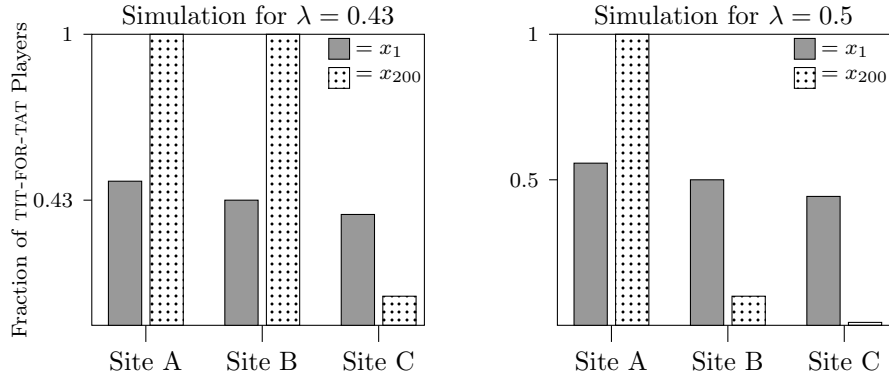


Figure 4: Simulations of the 3-Site Line after 200 Rounds; (left) $\lambda = 0.43$; (right) $\lambda = 0.5$.

Having established that the behavior of the middle site TIT-FOR-TAT population is not as trivial as the single-site analogue, we wish to determine how the long-term behavior of this demographic changes with respect λ . These figures indicate that for some smaller value of λ ($\lambda = 0.43$), the TIT-FOR-TAT population is able to thrive, and symmetrically, for some larger value of λ ($\lambda = 0.5$), the TIT-FOR-TAT population diminishes over time. This leads us to the main investigation of this thesis: for which values of λ do TIT-FOR-TAT players dominate the middle site, for which values do MEAN players dominate, and are there values in between where the two player types coexist?

3 The Ordering Lemma

An important fact in the analysis of the three-site model is that $a_n > b_n > c_n$ for all $n > 1$ if $a_1 > b_1 > c_1$. This fact, which we call the *Ordering Lemma*, is established by considering the derivatives of the increment function and its close relative

$$I_j(x, w, p) = \frac{x(1-x)(2wp + w - 1)}{2wp x + 4 + (1-w)(3p-x)}. \quad (18)$$

Note that whereas I is written as a function of only x in equation (16) because p depends on x in the context of the tournament and w is a parameter of the tournament, here we are treating p and w as independent variables.

Lemma 2. *Suppose that x , w , and p are positive numbers less than 1. Then the partial derivative of $I_j(x, w, p)$ with respect to p is positive.*

Proof. The quotient rule gives us

$$\begin{aligned} \frac{\partial I_j}{\partial p} &= \frac{2wx(1-x)(2wp x + 4 + (1-w)(3p-x))}{(2wp x + 4 + (1-w)(3p-x))^2} \\ &\quad - \frac{x(1-x)(2wp + w - 1)(2wx + 3(1-w))}{(2wp x + 4 + (1-w)(3p-x))^2} \\ &= \frac{2wx(1-x) - I_j(x, w, p)(2wx + 3(1-w))}{2wp x + 4 + (1-w)(3p-x)} \\ &= \frac{2wx + 3(1-w)}{2wp x + 4 + (1-w)(3p-x)} \left(\frac{2wx(1-x)}{2wx + 3(1-w)} - I_j(x, w, p) \right) \end{aligned}$$

The numerator of the leading factor is clearly positive since w and x are both in $(0, 1)$, and we have shown previously that the expression in the denominator is also positive, hence the leading factor is positive. By using the formula for $I_j(x, w, p)$, we find that the second term is positive only when

$$0 < \frac{2wx(1-x)}{2wx + 3(1-w)} - \frac{x(1-x)(2wp + w - 1)}{2wp x + 4 + (1-w)(3p-x)}.$$

Multiplying by the positive common denominator and dividing by the positive factor of $x(1-x)$, we have

$$\begin{aligned} 0 &< 2w(2wp x + 4 + (1-w)(3p-x)) - (2wp + w - 1)(2wx + 3 - 3w) \\ 0 &< 8w + 3(1-w)^2 \end{aligned}$$

which is true when $w \in (0, 1)$. □

Lemma 3. *The derivative of I_j with respect to x is greater than -1 when x , w , and p are positive numbers less than 1.*

Proof. The derivative of I_j with respect to x is

$$\frac{dI_j}{dx} = \frac{\partial I_j}{\partial x} + \frac{\partial I_j}{\partial p} \frac{\partial p}{\partial x}.$$

Using the Quotient Rule, the first term of this derivative is

$$\begin{aligned} \frac{\partial I_j}{\partial x} &= \frac{(1-2x)(2wp + w - 1)(2wp x + 4 + (1-w)(3p-x))}{(2wp x + 4 + (1-w)(3p-x))^2} \\ &\quad - \frac{x(1-x)(2wp + w - 1)(2wp - (1-w))}{(2wp x + 4 + (1-w)(3p-x))^2} \\ &= \frac{(1-2x)(2wp + w - 1)}{2wp x + 4 + (1-w)(3p-x)} - I_j \cdot \left(\frac{2wp + w - 1}{2wp x + 4 + (1-w)(3p-x)} \right) \\ &= \frac{1}{x(1-x)} ((1-2x)I_j - (I_j)^2). \end{aligned}$$

We now consider the second term of $\frac{dI_j}{dx}$, denoted by P so that $\frac{dI_j}{dx} = \frac{\partial I_j}{\partial x} + P$. We have from Lemma 2 that $\frac{\partial I_j}{\partial p}$ is positive, and since p is a weighted average of terms including x , $\frac{\partial p}{\partial x}$ is a positive constant, hence P must be positive.

In order to determine for which x we have $\frac{dI_j}{dx} > -1$, we consider solutions to the equation $-1 = \frac{dI_j}{dx}$:

$$-1 = \frac{1}{x(1-x)} ((1-2x)I_j - (I_j)^2) + P$$

Multiplying by $-x(1-x)$ and gathering the terms on the right-hand side yields a quadratic equation in I_j ,

$$0 = (I_j)^2 - (1-2x)I_j - x(1-x)(P+1).$$

Using the quadratic formula, we have that

$$\begin{aligned} I_j &= \left(\frac{1}{2}\right) (1-2x \pm \sqrt{(1-2x)^2 + 4x(1-x)(P+1)}) \\ &= \left(\frac{1}{2}\right) (1-2x \pm \sqrt{1+4x(1-x)P}) \end{aligned}$$

Since $P > 0$, the above formulation of I_j tells us

$$\begin{aligned} I_j &< \frac{1}{2}(1-2x-\sqrt{1}) & \text{or} & & I_j &> \frac{1}{2}(1-2x+\sqrt{1}) \\ I_j &< -x & \text{or} & & I_j &> 1-x \end{aligned}$$

It is clear from equation (8) that if $x_n \in [0, 1]$, then $x_{n+1} \in [0, 1]$, and so

$$0 \leq x + I_j(x) \leq 1 \implies -x \leq I_j(x) \leq 1-x.$$

Therefore it is not possible for $\frac{dI_j}{dx} = -1$ in the interval $(0, 1)$. \square

Ordering Lemma: *Suppose $a_1 > b_1 > c_1$. Then $p_A > p_B > p_C$ in all subsequent rounds, and $a_n > b_n > c_n$ when $n \geq 1$.*

Proof. We will prove the two results of this lemma separately. First, we show that the effective populations follow the ordering $p_A > p_B > p_C$ given that $a_n > b_n > c_n$. We will then show using mathematical induction that $a_n > b_n > c_n$ for all rounds n under the initial conditions established in (17).

The first assertion of this lemma is that the effective populations at each site exhibit the same ordering as the actual population fractions in round n . This can be seen by characterizing each of the effective populations as a weighted average of weighted averages. For example,

$$p_A = \left[\frac{a_n + qb_n}{1+q} \right] q + [a_n](1-q)$$

is a weighted average of the terms in brackets, as is

$$p_B = \left[q \left(\frac{qa_n + b_n + qc_n}{1 + 2q} \right) + (1 - q) \left(\frac{b_n + qa_n}{1 + q} \right) \right] q + \left[(1 - q)b_n + q \left(\frac{b_n + qc_n}{1 + q} \right) \right] (1 - q).$$

These two weighted averages have the same weights, namely q and $1 - q$, so we show that $p_B < p_A$ by demonstrating that both values to be averaged in p_A are greater than the corresponding values in p_B : that is we will establish that

$$(1 - q)b_n + q \left(\frac{b_n + qc_n}{1 + q} \right) < a_n \quad (19)$$

and

$$q \left(\frac{qa_n + b_n + qc_n}{1 + 2q} \right) + (1 - q) \left(\frac{b_n + qa_n}{1 + q} \right) < \frac{a_n + qb_n}{1 + q}. \quad (20)$$

Towards establishing inequality (19), we note that because c_n and b_n are both less than a_n , so is the weighted average

$$\frac{1}{1 + q}b_n + \frac{q}{1 + q}c_n = \frac{b_n + qc_n}{1 + q}.$$

Inequality (19) follows immediately because the left-hand side is the weighted average of terms that are each less than a_n . In a similar style, we prove inequality (20) by showing that the left-hand side is the weighted average of two terms, each less than the right-hand side. We first consider the term

$$\frac{b_n + qa_n}{1 + q} = \frac{1}{1 + q}b_n + \frac{q}{1 + q}a_n.$$

Since $q \in (0, 1)$, we have that $\frac{1}{1+q} > \frac{q}{1+q}$, and so this weighted average has a lesser weight on a_n than b_n . Since $a_n > b_n$, we can increase the value of the weighted average by reversing the weights, so

$$\frac{1}{1 + q}b_n + \frac{q}{1 + q}a_n < \frac{1}{1 + q}a_n + \frac{q}{1 + q}b_n = \frac{a_n + qb_n}{1 + q}.$$

We now consider the term

$$\frac{qa_n + b_n + qc_n}{1 + 2q} = \frac{q}{1 + 2q}a_n + \frac{1}{1 + 2q}b_n + \frac{q}{1 + 2q}c_n.$$

Since $c_n < b_n$,

$$\frac{q}{1+2q}a_n + \frac{1}{1+2q}b_n + \frac{q}{1+2q}c_n < \frac{q}{1+2q}a_n + \frac{1+q}{1+2q}b_n.$$

It is clear that $\frac{q}{1+2q} < \frac{1}{1+q}$ when $q \in (0, 1)$, and so we can construct a larger weighted average by replacing the coefficient of a_n from above with $\frac{1}{1+q}$ and adjusting the coefficient of b_n to be $\frac{q}{1+q}$. Therefore

$$\frac{q}{1+2q}a_n + \frac{1+q}{1+2q}b_n < \frac{1}{1+q}a_n + \frac{q}{1+q}b_n = \frac{a_n + qb_n}{1+q},$$

since the latter weighted average places a higher weight on a_n and $a_n > b_n$. Now we have established inequality (20), since the left-hand side is a weighted average of terms that are each less than the right-hand side.

We follow a similar proof for $p_C < p_B$ by considering

$$p_B = \left[q \left(\frac{qa_n + b_n + qc_n}{1+2q} \right) + (1-q) \left(\frac{b_n + qc_n}{1+q} \right) \right] q + \left[(1-q)b_n + q \left(\frac{b_n + qa_n}{1+q} \right) \right] (1-q)$$

and

$$p_C = \left[\frac{c_n + qb_n}{1+q} \right] q + [c_n](1-q),$$

and then constructing the corresponding pair of inequalities:

$$(1-q)b_n + q \left(\frac{b_n + qa_n}{1+q} \right) > c_n \tag{21}$$

and

$$q \left(\frac{qa_n + b_n + qc_n}{1+2q} \right) + (1-q) \left(\frac{b_n + qc_n}{1+q} \right) > \frac{c_n + qb_n}{1+q}. \tag{22}$$

We begin with inequality (21). The term

$$\frac{b_n + qa_n}{1+q} = \frac{1}{1+q}b_n + \frac{q}{1+q}a_n$$

is a weighted average of values each of which is greater than c_n , hence the average is greater than c_n as well.

We show that inequality (22) is true much as we did for inequality (20). Since $c_n < b_n$, the weighted average

$$\frac{b_n + qc_n}{1 + q} = \frac{1}{1 + q}b_n + \frac{q}{1 + q}c_n$$

is greater than the weighted averaged obtained by reversing the weights. Hence

$$\frac{1}{1 + q}b_n + \frac{q}{1 + q}c_n > \frac{q}{1 + q}b_n + \frac{1}{1 + q}c_n.$$

Using the fact that $a_n > b_n$, we now consider the first term of inequality (22)

$$\frac{qa_n + b_n + qc_n}{1 + 2q} > \frac{(1 + q)b_n + qc_n}{1 + 2q} = \frac{1 + q}{1 + 2q}b_n + \frac{q}{1 + 2q}c_n.$$

As before, we have that $\frac{1+q}{1+2q} < \frac{1}{1+q}$ when $q \in (0, 1)$, so it follows that

$$\frac{1 + q}{1 + 2q}b_n + \frac{q}{1 + 2q}c_n > \frac{q}{1 + q}b_n + \frac{1}{1 + q}c_n = \frac{qb_n + c_n}{1 + q}.$$

Hence the left-hand side of inequality (22) is a weighted average of terms that are each greater than the right-hand side, and so the inequality is true. Having established all four inequalities, we now have that $p_A > p_B > p_C$.

Now we address the assertion that $a_n > b_n > c_n$ for all rounds n . Since the initial conditions established in (17) have the ordering $a_1 > b_1 > c_1$, this ensures the base case is true. Now assume that $a_n > b_n > c_n$ for some $n \in \mathbb{N}$ and consider the ordering of a_{n+1} , b_{n+1} , and c_{n+1} . Let us write $a_{n+1} = a_n + I_A(a_n)$ and $b_{n+1} = b_n + I_B(b_n)$. Then $b_{n+1} > a_{n+1}$ if and only if $b_n - a_n > I_A(a_n) - I_B(b_n)$, which would require

$$-1 > \frac{I_A(a_n) - I_B(b_n)}{a_n - b_n}. \quad (23)$$

Because $p_A > p_B$ in round n and Lemma 2 states that the partial derivative of I_j with respect to p is positive, we have that $I_B(b_n) < I_A(b_n)$, and so

$$\frac{I_A(a_n) - I_B(b_n)}{a_n - b_n} > \frac{I_A(a_n) - I_A(b_n)}{a_n - b_n}. \quad (24)$$

Further, since I_A is a rational function with a non-zero denominator in $[0, 1]$, it is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Therefore the Mean Value Theorem tells us that there is some value, say x_0 , between a_n and b_n at which

$$\frac{I_A(a_n) - I_A(b_n)}{a_n - b_n} = \frac{dI_A(x_0)}{dx}.$$

In light of equations (23) and (24), this would mean that there is some point at which the derivative of I_A with respect to x is less than -1 , which contradicts Lemma 3. Therefore $a_{n+1} > b_{n+1}$. The proof for $b_{n+1} > c_{n+1}$ follows similarly. \square

4 The Net Lateral Increment Function

As mentioned in Section 2, we wish to determine conditions under which one of the player types in Site B will tend towards extinction. In order to further investigate this question, we will use the function $f(a, c) = I_A(a) + I_C(c)$, called the Net Lateral Increment Function, which calculates the net increment in the lateral sites, A and C. In this section we prepare for the discussion of the long-term behavior of the population in Site B by establishing some important facts about f in the initial round of the tournament.

Before beginning our analysis of the function f , we introduce the surface plot $f = 0$ in order to provide some visualization of this function's behavior. Note that f depends on the TIT-FOR-TAT population fraction in each of the three sites, and so we plot the level surface $f = 0$ as seen in Figure 5 in order to illustrate the regions in which f is positive or negative. For the rest of this section we will consider this function for

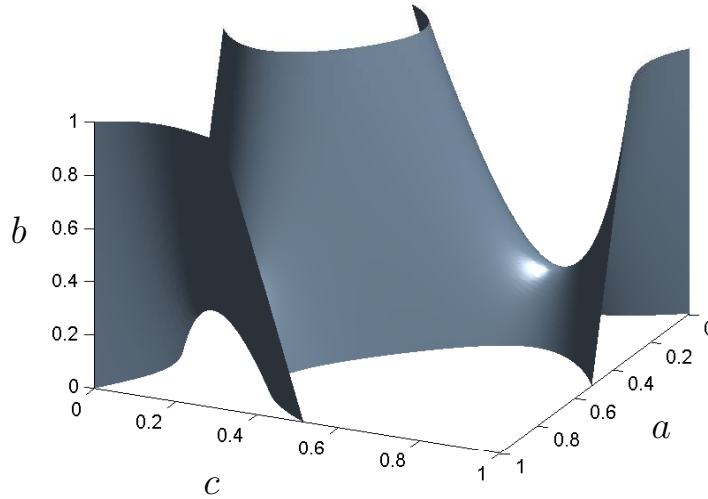


Figure 5: A graph of the level surface $f(a, b, c) = 0$. The region underneath the surface is where $f < 0$, and the region above is where $f > 0$.

a fixed value of b , namely $b_1 = \lambda$. For this reason, we can consider just a cross section of the above level surface in the b -dimension as seen in Figure 6.

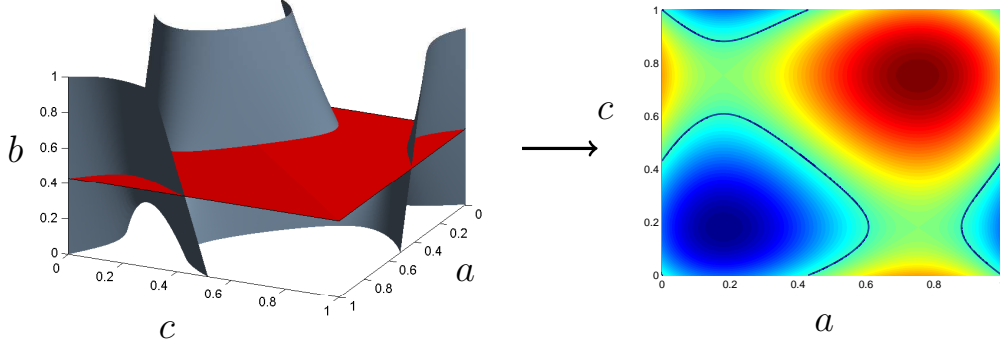


Figure 6: A cross section of f for $b = b_1 = \lambda$. The color map on the right illustrates the positive region of f in red and the negative region in blue.

Given the fact that $b = b_1 = \lambda$, both I_A and I_C simplify to the form

$$I(x) = \frac{x(1-x)(p-\lambda)}{px + 2 + 4\lambda + \lambda(3p-x)}.$$

We now consider the terms in this equation containing p for simplification. Using equation (13), we can write

$$px = \left(\frac{1+q-q^2}{1+q} \right) x^2 + \left(\frac{q^2}{1+q} \right) x\lambda,$$

and

$$\begin{aligned} \lambda(3p-x) &= \left(\frac{3(1+q-q^2)}{1+q} \right) x\lambda + \left(\frac{3q^2}{1+q} \right) \lambda^2 - x\lambda \\ &= \left(\frac{3+3q-q^2-1-q}{1+q} \right) x\lambda + \left(\frac{3q^2}{1+q} \right) \lambda^2 \\ &= \left(\frac{2+2q-q^2}{1+q} \right) x\lambda + \left(\frac{3q^2}{1+q} \right) \lambda^2. \end{aligned}$$

Substituting these formulations into the denominator of the increment yields

$$\begin{aligned} px + 2 + 4\lambda + \lambda(3p - x) &= 2 + 4\lambda + \left(\frac{1+q-q^2}{1+q}\right)x^2 + \left(\frac{3q^2}{1+q}\right)\lambda^2 \\ &\quad + \left(\frac{2(1+q-q^2)}{1+q}\right)x\lambda. \end{aligned} \quad (25)$$

Factoring $1+q-q^2/1+q$ from all terms on the right-hand side of equation (25) and simplifying what remains,

$$\begin{aligned} px + 2 + 4\lambda + \lambda(3p - x) &= \left(\frac{1+q-q^2}{1+q}\right)(x^2 + 2x\lambda + k) \\ \text{where } k &= 2 + 4\lambda + \frac{q^2(3\lambda^2 + 4\lambda + 2)}{1+q-q^2}. \end{aligned} \quad (26)$$

In the numerator of $I(x)$ we can again use equation (13) to write

$$\begin{aligned} p - \lambda &= \left(\frac{1+q-q^2}{1+q}\right)x + \left(\frac{q^2}{1+q}\right)\lambda - \lambda \\ &= \left(\frac{1+q-q^2}{1+q}\right)x + \left(\frac{-1-q+q^2}{1+q}\right)\lambda \\ &= \left(\frac{1+q-q^2}{1+q}\right)(x - \lambda). \end{aligned}$$

Hence the numerator of the increment is

$$x(1-x)(p-\lambda) = x(1-x)\left(\frac{1+q-q^2}{1+q}\right)(x-\lambda).$$

Substituting these simplifications into the increment yields

$$\begin{aligned} I(x) &= \frac{\left(\frac{1+q-q^2}{1+q}\right)x(1-x)(x-\lambda)}{\left(\frac{1+q-q^2}{1+q}\right)(x^2 + 2x\lambda + k)} \\ &= \frac{x(1-x)(x-\lambda)}{x^2 + 2x\lambda + k}. \end{aligned} \quad (27)$$

This formulation shows us that in the first round of the tournament, the function $f(a, c)$ has roots at

$$(0, 0), (\lambda, 0), (1, 0), (0, \lambda), (\lambda, \lambda), (\lambda, 1), (0, 1), (1, \lambda), (1, 1).$$

Because $I_A(x)$ and $I_C(x)$ are both negative when $x \in (0, \lambda)$, we know that $f(a, c) < 0$ when $a, c \in (0, \lambda)$. Similarly, because $I_A(x)$ and $I_C(x)$ are both positive when $x \in (\lambda, 1)$, we know that $f(a, c) > 0$ when both arguments are in $(\lambda, 1)$. This information is depicted in Figure 7.

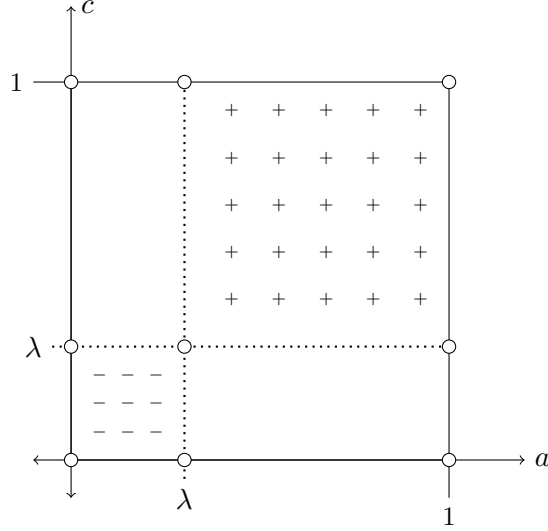


Figure 7: Circles indicate known points at which $f = 0$. The function is positive in the upper-right square, and negative in the lower-left square.

The following lemmas extend our knowledge of the region in which $f(a, c)$ is positive. Lemmas 4 and 5 establish that f is positive along particular line segments when λ is sufficiently small, and in Lemma 6 we prove that f is positive between them.

Lemma 4. *There is a number $\lambda_1 \geq \sqrt{1/5}$ such that $f(\lambda + \delta, \lambda - \delta) > 0$ in the first round of the tournament, provided that $\delta \in (0, \lambda]$ and $\lambda \in (0, \lambda_1)$.*

Proof. In the first round of the tournament we can write $f(\lambda + \delta, \lambda - \delta)$ as $I(\lambda + \delta) + I(\lambda - \delta)$. We begin by considering the value of $I(\lambda \pm \delta)$ using the formulation from equation (27).

$$I(\lambda + \delta) = \frac{(\lambda + \delta)(1 - \lambda - \delta)(\delta)}{(\lambda + \delta)^2 + 2\lambda(\lambda + \delta) + k} \quad I(\lambda - \delta) = \frac{(\lambda - \delta)(1 - \lambda + \delta)(-\delta)}{(\lambda - \delta)^2 + 2\lambda(\lambda - \delta) + k}$$

We see that the denominator of both terms is positive since $k > 0$, so the polarity of f depends on the combined numerator, denoted by N_I

$$\begin{aligned}
N_I &= (\lambda + \delta)(1 - \lambda - \delta)(\delta) \left[(\lambda - \delta)^2 + 2\lambda(\lambda - \delta) + k \right] \\
&\quad + (\lambda - \delta)(1 - \lambda + \delta)(-\delta) \left[(\lambda + \delta)^2 + 2\lambda(\lambda - \delta) + k \right] \\
&= \delta(\lambda + \delta - 2\lambda\delta - \lambda^2 - \delta^2) \left[3\lambda^2 + \delta^2 - 4\lambda\delta + k \right] \\
&\quad - \delta(\lambda - \delta + 2\lambda\delta - \lambda^2 - \delta^2) \left[3\lambda^2 + \delta^2 + 4\lambda\delta + k \right]
\end{aligned}$$

The similarity of the leading factors allows further simplification:

$$\begin{aligned}
N_I &= \delta \left((\lambda - \lambda^2 - \delta^2)(-8\lambda\delta) + (\delta - 2\lambda\delta)(6\lambda^2 + 2\delta^2 + 2k) \right) \\
&= \delta^2 \left((\lambda - \lambda^2 - \delta^2)(-8\lambda) + (1 - 2\lambda)(6\lambda^2 + 2\delta^2 + 2k) \right) \\
&= 2\delta^2 \left((1 - 2\lambda)k + (1 + 2\lambda)\delta^2 - \lambda^2 - 2\lambda^3 \right)
\end{aligned}$$

This is positive when $\delta > 0$ and

$$(1 - 2\lambda)k - \lambda^2 - 2\lambda^3 > 0.$$

We first note that when $\lambda < 1/2$, the factor $(1 - 2\lambda)$ is positive and $2\lambda^3 < \lambda^2$. Equation (26) tells us that $k > 2 + 4\lambda$, hence

$$\begin{aligned}
(1 - 2\lambda)k - \lambda^2 - 2\lambda^3 &> (1 - 2\lambda)(2 + 4\lambda) - \lambda^2 - 2\lambda^3 \\
&> (2 + 4\lambda - 4\lambda - 8\lambda^2) - \lambda^2 - \lambda^2 \\
&= 2 - 10\lambda^2.
\end{aligned}$$

Thus the numerator of $f(\lambda + \delta, \lambda - \delta)$ is positive when $2 - 10\lambda^2$ is positive, which happens when $\lambda < \sqrt{1/5}$. \square

Lemma 5. *There is a number $\lambda_2 \in (0, 1/2]$ such that if $\lambda \in (0, \lambda_2)$, the following statement is true in the first round of the tournament: there is a number $\alpha \in [2\lambda, 1)$ such that $f(\alpha, c) > 0$ when $c \in [0, 1]$.*

Proof. Since $I(x)$ as posed in equation (27) is a rational function with a non-zero denominator in $[0, 1]$, it is continuous on $[0, 1]$. Note that when $\lambda = 0$,

$$I(x) = \frac{x(1-x)(x-\lambda)}{x^2 + 2\lambda x + k} = \frac{x^2(1-x)}{x^2 + k}.$$

This function has roots at $x = 0$ and $x = 1$ and is positive for $x \in (0, 1)$. Therefore Extreme Value Theorem tells us that it achieves a positive maximum value at some $\alpha \in (0, 1)$, and so

$$I_{\max} + I_{\min} > 0, \tag{28}$$

where I_{\max} and I_{\min} denote the maximum and minimum values of $I(x)$ on $[0, 1]$, respectively. Because these critical points of $I(x)$ vary continuously with λ , as does the value of $I(x)$ at those points, the inequality in (28) remains true when $\lambda \in [0, 1]$ is sufficiently small. We have then that

$$f(\alpha, c) = I_{\max} + I_{\text{C}}(c) \geq I_{\max} + I_{\min} > 0$$

for all $c \in [0, 1]$. Further, because $\alpha - 2\lambda$ varies continuously with λ and is positive when $\lambda = 0$, we know that $\alpha > 2\lambda$ when λ is sufficiently small. \square

Lemma 6. *Suppose $\lambda \in (0, \min\{\lambda_1, \lambda_2\})$. In the first round of the tournament, the value of $f(a, b)$ is positive in the trapezoidal region of the ac -plane that is delimited by the lines $a + c = 2\lambda$, $a = \alpha$, $c = 0$, and $c = \lambda$, except at the point (λ, λ) .*

Proof. We demonstrate this fact by showing that all possible roots of the net lateral increment function $f(a, c)$ must lie outside the trapezoid and that the functional value inside the region is positive. Recall that

$$I(x) = \frac{x(1-x)(x-\lambda)}{x^2 + 2\lambda x + k},$$

where k is positive (and so the denominator is positive). This means that the function has exactly three roots. We can also use long division to determine that

$$\frac{x(1-x)(x-\lambda)}{x^2 + 2\lambda x + k} = -x + 3\lambda + 1 + \frac{-x(k + \lambda + 2\lambda(3\lambda + 1)) + k(3\lambda + 1)}{x^2 + 2\lambda x + k},$$

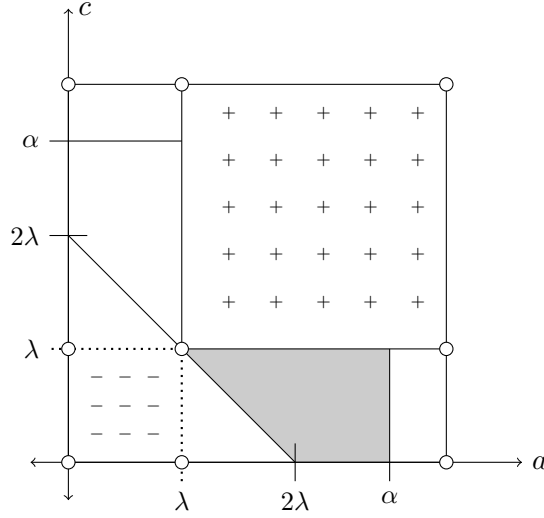


Figure 8: Incorporating information from Lemmas 4, 5, and 6 into Figure 7: the value of f is positive along the new line segments and in the shaded trapezoidal region.

therefore the line $y = -x + 3\lambda + 1$ is an oblique asymptote to the graph of $I(x)$ as $x \rightarrow \pm\infty$.

On the upper boundary of the region we have that $c = \lambda$ and $a > \lambda$, hence $I_C = 0$ and $I_A > 0$. This tells us then that $f(a, c) > 0$, and so we have established the result for the upper boundary of the region excluding the point (λ, λ) . We now consider the value of f along horizontal line segments at fixed, but arbitrary, $c^* \in [0, \lambda)$. The oblique asymptote of the increment function tells us that $f(a, c^*) = I_A(a) + I_C(c^*)$ is positive when a is sufficiently negative, and negative when a is sufficiently large. Further, since the increment function has exactly three roots, there can be at most three roots along the horizontal line $c = c^*$. Figure 9 depicts these facts (and more from the proof to follow).

We determine the locations of these roots by considering the polarity of f as we move from left to right along the horizontal line. When a is sufficiently negative, the functional value is positive. When we move into the region where $a \in [0, \lambda)$, we see

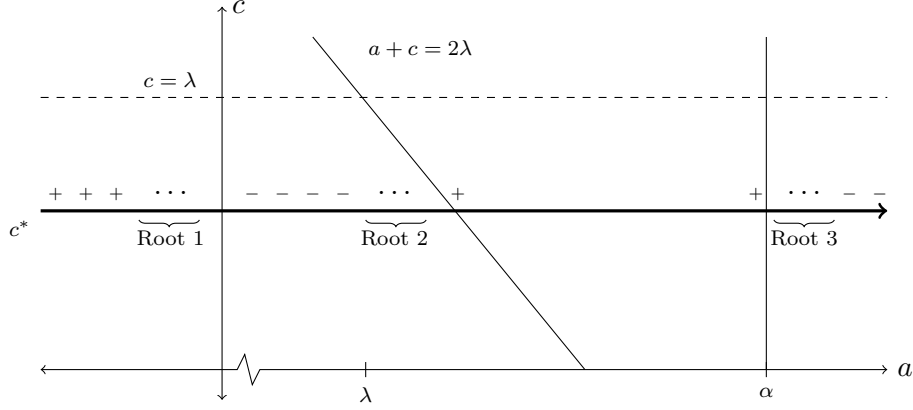


Figure 9: Polarity of the Net Lateral Increment Function as we move along fixed but arbitrary horizontal lines.

that $f(a, c^*) < 0$ since $I_A(a) \leq 0$ and $I_C(c^*) < 0$, hence one of the roots must occur to the left of the $[0, 1] \times [0, 1]$ square of the ac -plane. As we continue to move right, we encounter the line $a + c = 2\lambda$, which is the left border of the region. Lemma 4 tells us that the function is positive along this line, hence one of the roots lies between $a = \lambda$ and the line $a + c = 2\lambda$, which is still to the left of the region. We then move right through the region and hit the vertical line $a = \alpha$ guaranteed by Lemma 5, and we have that the function is positive here as well. As we continue moving right, the oblique asymptote dictates that the function eventually become negative again, and so one the roots occurs to the right of the vertical line $a = \alpha$. Hence all three roots exists outside of the region regardless of the choice of c^* , and so the function $f(a, c)$ does not vanish inside the region. \square

In the previous lemma we established a large region in which $f(a, c)$ is positive in the first round of the tournament when $b = \lambda$. The next lemma guarantees that f remains positive in this region if b increases.

Lemma 7. *The partial derivative of f with respect to b is positive.*

Proof. The value of b enters f through p , so we apply the Chain Rule:

$$\frac{\partial f}{\partial b} = \frac{\partial I_A}{\partial p_A} \frac{\partial p_A}{\partial b} + \frac{\partial I_C}{\partial p_C} \frac{\partial p_C}{\partial b}.$$

The factors of $\frac{\partial I_A}{\partial p_A}$ and $\frac{\partial I_C}{\partial p_C}$ are positive by virtue of Lemma 2, and equations (13) and (14) give us that $\frac{\partial p_A}{\partial b} = \frac{\partial p_C}{\partial b} = \frac{q}{1+q} > 0$. Hence the partial derivative of f with respect to b is positive, and so the region where $f > 0$ described in Lemma 6 remains positive as b increases. \square

Lemma 8. *Suppose $a_1 > b_1 > c_1$, and $f(a_n, c_n) > 0$. Then $I_A(a_n) > 0$.*

Proof. As seen in (16), the polarity of the increment function $I(x)$ is determined solely by the factor $(p - \lambda)$ in its numerator. Since $f(a_n, c_n) > 0$, it must be that one of $I_A(a_n)$ or $I_C(c_n)$ is positive. If it were true that $I_A(a_n) \leq 0 < I_C(c_n)$, the respective polarity-controlling factors of $I_A(a_n)$ and $I_C(c_n)$ would exhibit the same ordering, i.e.

$$(p_A - \lambda) < (p_C - \lambda) \Rightarrow p_A < p_C.$$

However, since $a_1 > b_1 > c_1$, the Ordering Lemma tells us that $p_A > p_B > p_C$ for all rounds n . This is a contradiction, so we conclude that $I_A(a_n)$ must be positive. \square

5 Monotonic Increase of b_n for Sufficiently Small λ

The simulations presented in Section 2 show that either player type can dominate Site B in the long run. In this section we prove that TIT-FOR-TAT players will dominate the middle site in the long run ($b_n \rightarrow 1$ as $n \rightarrow \infty$) for sufficiently small λ . Our proof begins by showing that the increment $I_B(b_n) > 0$ in all rounds $n \geq 1$. Since $b_n \leq 1$ by definition, the sequence $\{b_n\}_{n=1}^{\infty}$ is bounded and monotonic, hence it must converge. We conclude by demonstrating that the limit value of this sequence is 1.

Theorem 1. *Suppose $\lambda \in (0, \min\{\lambda_1, \lambda_2\})$, and a_1 , b_1 , and c_1 are initialized as in Section 2. Then $\{b_n\}_{n=1}^{\infty}$ is a monotonically increasing sequence.*

Proof. As seen in (16), the polarity of $I_B(b_n)$ is determined solely by the factor of $(p_B - \lambda)$ in its numerator. Specifically, $I_B(b_n) > 0$ when $p_B > \lambda$. Using equation (15), we can rewrite $p_B > \lambda$ as

$$\left(\frac{q^2 + 2q^3 - q^4}{1 + 3q + 2q^2} \right) (a_n + c_n) + \left(\frac{1 + 3q - 4q^3 + 2q^4}{1 + 3q + 2q^2} \right) b_n > \lambda.$$

Multiplying both sides by $(1+q)(1+2q) = 1+3q+2q^2$ and moving the term containing b_n to the right-hand side yields

$$(q^2 + 2q^3 - q^4)(a_n + c_n) > (1 + 3q + 2q^2)\lambda - (1 + 3q - 4q^3 + 2q^4)b_n. \quad (29)$$

We consider this inequality in the first round when $b_1 = \lambda$.

$$\begin{aligned} (q^2 + 2q^3 - q^4)(a_1 + c_1) &> (1 + 3q + 2q^2)\lambda - (1 + 3q - 4q^3 + 2q^4)\lambda \\ (q^2 + 2q^3 - q^4)(a_1 + c_1) &> (2q^2 + 4q^3 - 2q^4)\lambda \\ a_1 + c_1 &> 2\lambda \end{aligned}$$

In the first round we have that $a_1 = \lambda + \varepsilon(1 - \lambda)$ and $c_1 = \lambda - \varepsilon\lambda$ from (17), so

$$\begin{aligned} a_1 + c_1 &= \lambda + \varepsilon(1 - \lambda) + \lambda - \varepsilon\lambda \\ &= 2\lambda - 2\varepsilon\lambda + \varepsilon \\ &= 2\lambda + \varepsilon(1 - 2\lambda). \end{aligned}$$

Since $\lambda < 1/2$, the quantity $1 - 2\lambda$ is positive, hence

$$a_1 + c_1 > 2\lambda.$$

Since the inequality holds in the first round, we have that $p_B > \lambda$ and $b_2 > b_1$.

Additionally, note that $a_1 > \lambda > c_1$. Recalling the role of α from Lemma 5, we see that if $a_1 \leq \alpha$, the point (a_1, c_1) lies in the region of the ac -plane where, according to Lemma 6, the value of the net lateral increment function, f , is positive. We wish to consider how the point (a_n, c_n) moves through the plane from one round to the next, and so we formulate its movement as the vector $\langle I_A(a_n), I_C(c_n) \rangle$. Lemma 8 then tells us that $I_A(a_1) > 0$, so the vector $\langle I_A(a_1), I_C(c_1) \rangle$ points to the right. Since $f(a_1, c_1) > 0$, we know that the vector cannot have a slope less than -1, otherwise this would imply that $|I_C(c_1)| > I_A(a_1)$ and so $f < 0$. Since (a_1, c_1) lies above the line $a + c = 2\lambda$ and the vector $\langle I_A(a_1), I_C(c_1) \rangle$ points to the right along a line with slope greater than -1, we have that (a_2, c_2) also lies above the line $a + c = 2\lambda$. In fact, because $b_2 > b_1$, the inequality in (29) has become easier to satisfy: that inequality says that the point (a_n, c_n) must lie above the line,

$$(q^2 + 2q^3 - q^4)(a_n + c_n) = (1 + 3q + 2q^2)\lambda - (1 + 3q - 4q^3 + 2q^4)b_n, \quad (30)$$

in order for the increment in Site B to be positive. As b_n increases, we see that the a -intercept of this line decreases. In short, in the transition from the first to the second round of the tournament, the number b_n increases, the point (a_n, c_n) moves to the right, and the line from equation (30), which is where $p_B = \lambda$, slides left. If $a_2 \leq \alpha$, the preceding argument is reapplied to show that $a_3 > a_2$, $b_3 > b_2$, and the point (a_3, c_3) lies above the line where $p_B = \lambda$. The argument continues to apply until $a_n > \alpha$, at which point we are no longer certain that $f(a_n, c_n) > 0$. This process is demonstrated in Figure 10.

Now we argue that b_n is monotonically increasing when $a_n > \alpha$ (including the case when this is true of a_1). In the first such round we have that $a_n > \lambda$ and $b_n \geq \lambda$, and so $p_A > \lambda$. It follows that $I_A(a_n) > 0$, and so (a_{n+1}, c_{n+1}) is to the right of (a_n, c_n)

in the ac -plane. Further, the point (a_n, c_n) lies above the line $p_B = \lambda$ because the a -intercept of the line is at

$$\begin{aligned} a &= \frac{(1 + 3q + 2q^2)\lambda - (1 + 3q - 4q^3 + 2q^4)b_n}{(q^2 + 2q^3 - q^4)} \\ &\leq \frac{(1 + 3q + 2q^2)\lambda - (1 + 3q - 4q^3 + 2q^4)b_1}{(q^2 + 2q^3 - q^4)} \\ &= 2\lambda < \alpha \end{aligned}$$

by virtue of Lemma 5. It follows that $b_{n+1} > b_n$. Further, because $p_A > \lambda$ in round n , and both a and b have increased from round n to round $n + 1$, we know that $p_A > \lambda$ in round $n + 1$ as well. The point (a_{n+1}, c_{n+1}) lies above the line $p_B = \lambda$, and this argument can be applied in round $n + 1$. The conclusion of monotonicity follows. \square

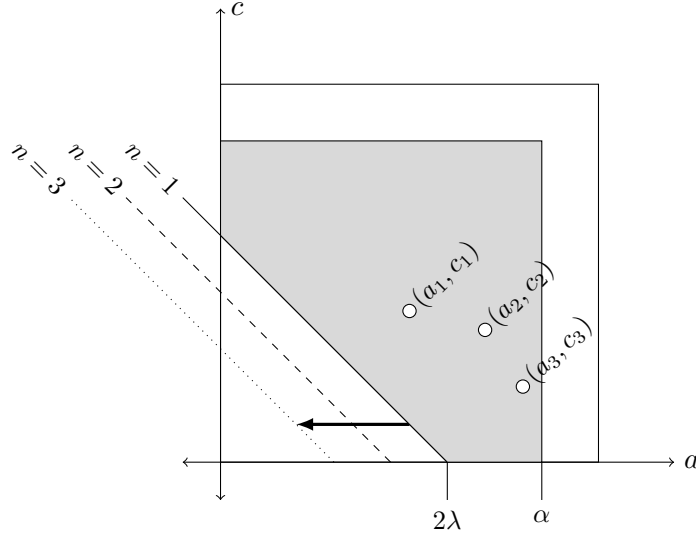


Figure 10: Schematic representation of the motion of (a_n, c_n) through the ac -plane. The line at which $p_B = \lambda$ moves left each time b_n increases.

Theorem 2. Suppose $\lambda \in (0, \min\{\lambda_1, \lambda_2\})$, and a_1 , b_1 , and c_1 are initialized as in Section 2. Then $\lim_{n \rightarrow \infty} b_n = 1$.

Proof. Because b_n is increasing monotonically and is bounded above by 1, it must converge, say $b_n \rightarrow L$. Consequently,

$$\begin{aligned}
0 &= L - L \\
&= \lim_{n \rightarrow \infty} b_{n+1} - \lim_{n \rightarrow \infty} b_n \\
&= \lim_{n \rightarrow \infty} (b_{n+1} - b_n) \\
&= \lim_{n \rightarrow \infty} I_B(b_n)
\end{aligned}$$

Since the increment function is continuous on $[0, 1]$,

$$0 = I_B \left(\lim_{n \rightarrow \infty} b_n \right) = I_B(L) \quad (31)$$

In principle, equation (31) could be true because $p_B \rightarrow \lambda$ as $b_n \rightarrow L$, so our proof relies heavily on the fact, established below, that there is a number p_δ such that $p_B \geq p_\delta > \lambda$ in all rounds of the tournament.

We wish to construct a δ such that the line $a + c = 2\lambda + \delta$ remains below the point (a_n, c_n) for all rounds $n \geq 1$. First, since $\lambda < \lambda_2$, Lemma 5 guarantees a value $\alpha \in [2\lambda, 1)$ such that $f(\alpha, c) > 0$ when $c \in [0, 1]$. The net lateral increment function is uniformly continuous on $[0, 1] \times [0, 1]$, so there exists some $\delta_1 > 0$ such that $f(\alpha + \delta, c) > 0$ when $\delta \in (0, \delta_1)$ and $c \in [0, 1]$. Further, in the first round of the tournament we know that

$$a_1 + c_1 = 2\lambda + \varepsilon(1 - 2\lambda),$$

where $\varepsilon(1 - 2\lambda) > 0$ since $\lambda < 1/2$. Let us define $\delta_2 = \varepsilon(1 - 2\lambda)$, and then let $\delta = 0.5\min\{\delta_1, \delta_2\}$. Then the segment of the line $a + c = 2\lambda + \delta$ on which $c \in [0, \lambda]$ lies in the region where $f(a, c) > 0$, and in the first round we have

$$a_1 + c_1 > 2\lambda + \delta.$$

In Theorem 1 we saw that when (a_1, c_1) begins above the line $a + c = 2\lambda$, the point (a_n, c_n) remains above that line in all subsequent rounds due to the fact that the

vector $\langle I_A(a_n), I_C(c_n) \rangle$ points along a line with slope greater than -1 until $a_n > \alpha$. Since (a_1, c_1) starts above the line $a + c = 2\lambda + \delta$ as well, the slope argument applies, and in fact will hold until $a_n > \alpha + \delta$. We know from Theorem 1 that when $a_n > \alpha$, $I_B(b_n) > 0$ for all subsequent rounds. In the present case where $a_n > \alpha + \delta$, we can apply this same argument to determine that the point (a_n, c_n) remains above the line $a + c = 2\lambda + \delta$ for all rounds $n \geq 1$.

We know that the points on the line $a + c = 2\lambda$ correspond to $p_B = \lambda$ in the first round, so we now consider a similar correspondence for the rounds when $a_n + c_n \geq 2\lambda + \delta$. We do so by following the derivation of equation (29) in reverse, and so we begin by multiplying both sides of the inequality by $q^2(1 + 2q - q^2)$, which is positive since $q \in (0, 1)$,

$$q^2(1 + 2q - q^2)(a_n + c_n) \geq 2q^2(1 + 2q - q^2)\lambda + q^2(1 + 2q - q^2)\delta.$$

Using the fact that $b_1 = \lambda$,

$$q^2(1 + 2q - q^2)(a_n + c_n) \geq (1 + 3q + 2q^2)\lambda - (1 + 3q - 4q^3 + 2q^4)b_1 + q^2(1 + 2q - q^2)\delta.$$

We then divide both sides by $(1 + q)(1 + 2q) = 1 + 3q + 2q^2$ and note that $b_n > b_1$ for all $n > 1$ by virtue of Theorem 1, which yields

$$\frac{q^2 + 2q^3 - q^4}{1 + 3q + 2q^2}(a_n + c_n) \geq \lambda - \frac{1 + 3q - 4q^3 + 2q^4}{1 + 3q + 2q^2}b_n + \frac{q^2(1 + 2q - q^2)}{1 + 3q + 2q^2}\delta.$$

Moving the term containing b_n to the left-hand side gives us the definition for p_B , hence

$$p_B \geq \lambda + \frac{q^2(1 + 2q - q^2)}{1 + 3q + 2q^2}\delta.$$

We define p_δ to be the value on the right-hand side of this inequality, and so $p_B \geq p_\delta > \lambda$ for all rounds $n \geq 1$.

Now that we have bounded p_B away from λ , we know that the increment must converge in the interval $(\lambda, 1]$. We show that $L = 1$ by bounding the increment below by a quadratic function with roots at 0 and 1 that is positive in $(0, 1)$, and so the

increment can only converge to 0 if the quadratic function does as well. We consider the definition of the increment and use the fact that $p_B \geq p_\delta$ to show that

$$\begin{aligned} I_B(b_n) &= (p_B - \lambda) \frac{b_n(1 - b_n)}{p_B b_n + 2 + 4\lambda + \lambda(3p_B - b_n)} \\ &\geq \left(\frac{q^2(1 + 2q - q^2)}{(1 + q)(1 + 2q)} \delta \right) \frac{b_n(1 - b_n)}{p_B b_n + 2 + 4\lambda + \lambda(3p_B - b_n)}. \end{aligned}$$

Since $p_B, b_n \in [0, 1]$, the denominator of the second factor is no more than $1 + 2 + 4\lambda + 3\lambda = 3 + 7\lambda$, so

$$I_B(b_n) \geq \left(\frac{q^2(1 + 2q - q^2)}{(1 + q)(1 + 2q)} \delta \right) \frac{b_n(1 - b_n)}{3 + 7\lambda}.$$

That is, $I_B(b_n) \geq A\delta b_n(1 - b_n)$ where

$$A = \frac{q^2(1 + 2q - q^2)}{(1 + q)(1 + 2q)(3 + 7\lambda)}$$

is a positive constant. Since $0 \leq A\delta b_n(1 - b_n) \leq I_B(b_n)$ when $p_B > p_\delta$, the Squeeze Theorem tells us that this quadratic expression must converge to 0 wherever the increment does. Because b_n is increasing and the quadratic expression only has roots at 0 and 1, it must be that $\lim_{n \rightarrow \infty} b_n = L = 1$. \square

6 Monotonic Decrease of b_n for Sufficiently Large λ

In the previous section, we showed that for sufficiently small values of λ , the middle site TIT-FOR-TAT population, b_n , tends to 1 as $n \rightarrow \infty$. In this section we establish the symmetric result: for sufficiently large λ , b_n will tend to 0 as $n \rightarrow \infty$. We do so by first adjusting several lemmas from Section 4, and then proceeding as in Section 5.

Lemma 4b. *Suppose $\lambda \in (1/2, 1)$. Then $f(\lambda + \delta, \lambda - \delta) < 0$ in the first round of the tournament, provided that $\delta \in (0, 1 - \lambda]$.*

Proof. We begin by noting that because $\lambda \in (1/2, 1]$, the line formed by the points $(\lambda + \delta, \lambda - \delta)$ for $\delta > 0$ intersects the right-hand edge of the $[0, 1] \times [0, 1]$ square in the ac -plane. This intersection occurs precisely when $\delta = 1 - \lambda$, and so we consider the mathematically useful hypothesis that $\delta \leq 1 - \lambda$ since it has no effect on the model itself. As in Lemma 4 we consider the numerator of $f(\lambda + \delta, \lambda - \delta)$,

$$2\delta^2[(1 + 2\lambda)\delta^2 + k(1 - 2\lambda) - \lambda^2 - 2\lambda^3],$$

which is unchanged from the derivation in the Lemma 4. For large lambda, specifically $\lambda > 1/2$, we have that $(1 - 2\lambda) < 0$. Combined with the fact that $k > 2 + 4\lambda$ by its definition in equation (27), it follows from $\delta \leq 1 - \lambda$ that

$$\begin{aligned} & (1 + 2\lambda)\delta^2 + k(1 - 2\lambda) - \lambda^2 - 2\lambda^3 \\ & \leq (1 + 2\lambda)(1 - \lambda)^2 + (2 + 4\lambda)(1 - 2\lambda) - \lambda^2 - 2\lambda^3 \\ & = (1 + 2\lambda)(1 - 2\lambda + \lambda^2) + 2 - 4\lambda + 4\lambda - 8\lambda^2 - \lambda^2 - 2\lambda^3 \\ & = 1 - 2\lambda + \lambda^2 + 2\lambda - 4\lambda^2 + 2\lambda^3 + 2 - 9\lambda^2 - 2\lambda^3 \\ & = 3 - 12\lambda^2, \end{aligned}$$

which is negative when $\lambda > 1/2$. Hence $f(\lambda + \delta, \lambda - \delta) < 0$ in the first round. \square

Lemma 5b. *There is a number $\lambda_3 \in [1/2, 1)$ such that the following statement is true in the first round of the tournament if $\lambda \in (\lambda_3, 1)$: there is a number $\gamma \in (0, 2\lambda - 1]$ such that $f(a, \gamma) < 0$ when $a \in [0, 1]$.*

Proof. As in Lemma 5, we begin by examining the function $I(x)$ when $\lambda = 1$, so

$$I(x) = \frac{x(1-x)(x-\lambda)}{x^2 + 2\lambda x + k} = \frac{-x(1-x)^2}{x^2 + 2x + k}.$$

This function has roots at 0 and 1, and it is negative when $x \in (0, 1)$. This means that $I(x)$ achieves a negative minimum value for some $\gamma \in (0, 1)$, and that $I_{\max} + I_{\min} < 0$. As stated in Lemma 5, the uniform continuity of $I(x)$ tells us that the critical points vary continuously with λ as do the values of $I(x)$ at these points. This means the inequality remains true when $\lambda \in [0, 1]$ is sufficiently large, and that the value of $f(a, \gamma)$ is negative when $a \in [0, 1]$ since

$$f(a, \gamma) = I_A(a) + I_{\min} \leq I_{\max} + I_{\min} < 0.$$

Further, because $(2\lambda - 1) - \gamma$ varies continuously with λ and is positive when $\lambda = 1$, we know that $\gamma < 2\lambda - 1$ when λ is sufficiently large. \square

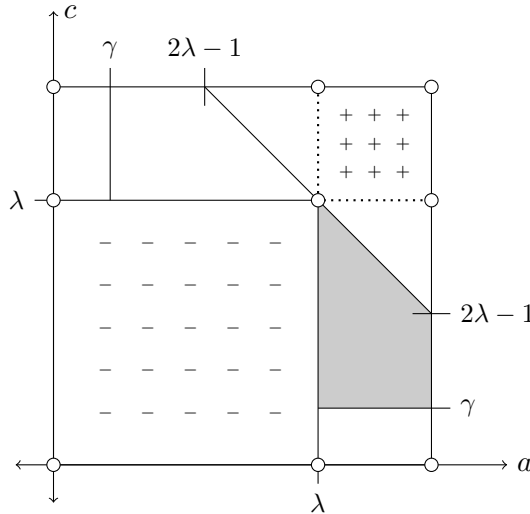


Figure 11: Incorporating information from Lemmas 4b, 5b, and 6b: the value of f is negative along the line segments $c = \gamma$, $a + c = 2\lambda$, $a = \lambda$, and inside the shaded, trapezoidal region (except at the point (λ, λ)).

Lemma 6b. *Suppose $\lambda \in (\max\{1/2, \lambda_3\}, 1)$. In the first round of the tournament, the value of $f(a, c)$ is negative in the trapezoidal region of the ac -plane that is delimited by the lines $a + c = 2\lambda$, $a = \lambda$, $a = 1$, and $c = \gamma$, except at the point (λ, λ) .*

Proof. As in Lemma 6, we demonstrate this fact by fixing arbitrary values of $a^* \in (\lambda, 1]$, showing that all three roots of f lie outside the region, and demonstrating that the value is negative on the interior. Recall that

$$I(x) = \frac{x(1-x)(x-\lambda)}{x^2 + 2\lambda x + k}.$$

In Lemma 6 we determined that this function only has three roots and has the oblique asymptote $y = -x + 3\lambda + 1$ as $x \rightarrow \pm\infty$. On the left-most boundary of the region, we have that $a = \lambda$ and $c < \lambda$, hence $I_A = 0$ and $I_C < 0$, so $f(a, c) < 0$. This establishes the result on the left-most boundary except for the point (λ, λ) . We now consider the value of f along vertical line segments for fixed but arbitrary $a^* \in (\lambda, 1]$. The oblique asymptote tells us that $f(a^*, c) = I_A(a^*) + I_C(c)$ is positive when c is sufficiently negative, and negative when c is sufficiently positive. As in Lemma 6, we locate the three roots of the equation $f(a^*, c)$ by moving from bottom to top and observing the polarity of f in certain regions.

When c is sufficiently negative, the oblique asymptote tells us that f is positive. We next encounter the line $c = \gamma$, and Lemma 5b tells us that $f(a^*, \gamma) < 0$, hence the first root must be below the line $c = \gamma$. Lemma 4b gives us that f is again negative on the line $a + c = 2\lambda$, which is the upper boundary of the region. We then cross into the region where $c \in (\lambda, 1]$ and $a^* > \lambda$, so it must be that $f(a^*, c)$ is positive in this region and another root occurs just above the line $a + c = 2\lambda$. Finally, as c becomes sufficiently positive, the oblique asymptote dictates that f turns negative, and so the final root occurs above the line $c = 1$. Therefore all three roots exist outside the region, and since a^* was chosen arbitrarily, we have that f does not vanish inside. \square

Lemma 8b. *Suppose $a_1 > b_1 > c_1$ and $f(a_n, c_n) < 0$. Then $I_C(c_n) < 0$.*

Proof. We know that at least one of $I_A(a_n)$ and $I_C(c_n)$ is negative because $f(a_n, c_n) < 0$. Following the proof from Lemma 8, the assumption that $I_A < 0 < I_C$ implies that $p_A < \lambda < p_C$, which contradicts the Ordering Lemma. Therefore it must be that $I_C(c_n) < 0$. \square

Theorem 1b. *Suppose $\lambda \in (\max\{1/2, \lambda_3\}, 1)$, and a_1 , b_1 , and c_1 are initialized as in Section 2. Then b_n decreases monotonically.*

Proof. Recall that the polarity of $I_B(x)$ is determined solely by the factor of $(p_B - \lambda)$ in its numerator, so that $I_B(b_n) < 0$ precisely when $p_B < \lambda$. Using the derivation of equation (29), but reversing the inequality so that $I_B(b_n) < 0$, we find that $I_B(b_n)$ is negative when

$$(q^2 + 2q^3 - q^4)(a_n + c_n) < (1 + 3q + 2q^2)\lambda - (1 + 3q - 4q^3 + 2q^4)b_n.$$

In this first round we have that $b_1 = \lambda$, so this inequality reduces to $a_1 + c_1 < 2\lambda$. This is satisfied since

$$\begin{aligned} a_1 + c_1 &= \lambda + \varepsilon(1 - \lambda) + \lambda - \varepsilon\lambda \\ &= 2\lambda + \varepsilon(1 - 2\lambda), \end{aligned}$$

and $\lambda > 1/2$. Hence $b_2 < b_1$.

It is clear from the initialization that (a_1, c_1) lies in the region defined in Lemma 6b where $f(a, c) < 0$, so Lemma 8b tells us that $I_C(c_1) < 0$, and hence the vector $\langle I_A(a_1), I_C(c_1) \rangle$ points downward in the ac -plane. In fact, since $f(a_1, c_1) < 0$, the vector must point along a line with slope less than -1 , and so the point (a_2, c_2) also lies below the line $a + c = 2\lambda$. We can repeat this process until $c_n < \gamma$, after which it is not certain that $f(a, c) < 0$.

We now show that b_n is monotonically decreasing even when $c_n < \gamma$, including the case when this is true for $n = 1$. Since $c_n < \lambda$ and $b_n \leq \lambda$, we know that $p_C < \lambda$. It follows that $I_C(c_n) < 0$, so (a_{n+1}, c_{n+1}) lies below (a_n, c_n) in the ac -plane. We

also have from Lemma 5b that $\gamma < 2\lambda - 1$, and so the point (a_n, c_n) lies below the intersection of the lines $a + c = 2\lambda$ and $a = 1$. Hence (a_{n+1}, c_{n+1}) lies below the line as well. Consequently, $b_{n+1} < b_n$, and since both b and c have decreased from round n to round $n + 1$, this argument can be reapplied. The conclusion of monotonicity follows. \square

Theorem 2b. *Suppose $\lambda \in (\max\{1/2, \lambda_3\}, 1)$ and a_1, b_1 , and c_1 are initialized as in Section 2. Then $\lim_{n \rightarrow \infty} b_n = 0$.*

Proof. Because b_n is decreasing monotonically and is bounded below by 0, it must converge to some value $L \in [0, \lambda)$. We saw in Theorem 2 that in order for $\{b_n\}$ to converge, the increment $I_B(b_n)$ must converge to 0. It could again be the case that this is true because $p_B \rightarrow \lambda$, so we begin by showing that there exists some number p_δ such that $p_B \leq p_\delta < \lambda$ in all rounds of the tournament.

As in Theorem 2, we wish to construct a δ such that the line $a + c = 2\lambda - \delta$ remains above the point (a_n, c_n) for all rounds $n \geq 1$. First, since $\lambda > \lambda_3$, Lemma 5b guarantees a value $\gamma \in (0, 2\lambda - 1]$ such that $(a, \gamma) < 0$ when $a \in [0, 1]$. We have noted before that the net lateral increment function is uniformly continuous on $[0, 1] \times [0, 1]$, so there exists some $\delta_1 > 0$ such that $f(a, \gamma - \delta) < 0$ when $\delta \in (0, \delta_1)$ and $a \in [0, 1]$. Second, in the first round of the tournament we have seen that

$$a_1 + c_1 = 2\lambda + \varepsilon(1 - 2\lambda),$$

where $\varepsilon(1 - 2\lambda) < 0$ since $\lambda > 0.5$. Let us define $\delta_2 = |\varepsilon(1 - 2\lambda)|$, and then $\delta = 0.5\min\{\delta_1, \delta_2\}$. Then the segment of the line $a + c = 2\lambda - \delta$ on which $a \in [2\lambda - 1, 1]$ lies in the region where $f(a, c) < 0$, and in the first round we have

$$a_1 + c_1 < 2\lambda - \delta.$$

As in Theorem 2, we see that because the point (a_1, c_1) starts below the line $a + c = 2\lambda - \delta$ and the vector $\langle I_A(a_n), I_C(c_n) \rangle$ has a slope less than -1 while $c_n \geq \gamma$, the point

(a_n, c_n) will remain below the line for all rounds until $c_n < \gamma$. In fact, this argument holds until $c_n < \gamma - \delta$, at which point we can use the argument from Theorem 1b for when $c_n < \gamma$ to determine that the point (a_n, c_n) remains below the line $a + c = 2\lambda - \delta$ for all rounds $n \geq 1$.

We now wish to find a relationship between the region $a_n + c_n \leq 2\lambda - \delta$ and p_B , and the derivation is nearly equivalent to the one seen in Theorem 2. The final result of this derivation tells us that

$$a_n + c_n \leq 2\lambda - \delta \iff p_B \leq \lambda - \frac{q^2(1 + 2q - q^2)}{1 + 3q + 2q^2}\delta.$$

We take this quantity on the right to be p_δ , and so $p_B \leq p_\delta < \lambda$ for all rounds $n \geq 1$.

Now that we have bounded p_B away from λ , we know that the increment must converge in the interval $[0, \lambda)$. As in Theorem 2, we show that $L = 0$ by bounding the increment with a quadratic function (in this case, it will be bounded above) of the form $-A\delta b_n(1 - b_n)$. The Squeeze Theorem tells us then that the increment can only converge to 0 if the quadratic function does as well, and this only occurs at $b_n = \{0, 1\}$. We consider the form of the increment function

$$\begin{aligned} I_B(b_n) &= (p_B - \lambda) \frac{b_n(1 - b_n)}{p_B b_n + 2 + 4\lambda + \lambda(3p_B - b_n)} \\ &\leq \left(-\frac{q^2(1 + 2q - q^2)}{(1 + q)(1 + 2q)}\delta \right) \frac{b_n(1 - b_n)}{p_B b_n + 2 + 4\lambda + \lambda(3p_B - b_n)}. \end{aligned}$$

Since $p_B, b_n \in [0, 1]$, the denominator of the second factor is at least $2 + 3\lambda - \lambda = 2(1 + \lambda)$, so

$$I_B(b_n) \leq -\left(\frac{q^2(1 + 2q - q^2)}{(1 + q)(1 + 2q)}\delta \right) \frac{b_n(1 - b_n)}{2(1 + \lambda)}.$$

That is, $I_B(b_n) \leq -A\delta b_n(1 - b_n)$ where

$$A = \frac{q^2(1 + 2q - q^2)}{2(1 + q)(1 + 2q)(1 + \lambda)}$$

is a positive constant. Since $I_B(b_n) \leq -A\delta b_n(1 - b_n) \leq 0$, the Squeeze Theorem tells us that the quadratic expression must converge to 0 wherever the increment does. Because

b_n is decreasing and the quadratic expression only has roots at 0 and 1, it must be that

$$\lim_{n \rightarrow \infty} b_n = L = 0. \quad \square$$

7 Analysis of Nontrivial Equilibria

In the previous sections we established that if λ is sufficiently small or sufficiently large in $[0, 1]$, one player type or the other is driven to extinction in the middle site. A natural question is whether middle values of λ allow for a steady-state solution in which $\lim_{n \rightarrow \infty} b_n \notin \{0, 1\}$ exists, and if so, if it is accessible given the initialization of the system. This leads us to look for equilibria, which are points (a, b, c) at which $(I_A, I_B, I_C) = (0, 0, 0)$, and we have seen previously that the increment $I(x) = 0$ if and only if $x = 0$, $x = 1$, or $p = \lambda$. Since we are looking for equilibria (a, b, c) in which b is neither 0 nor 1, it must be that $p_B = \lambda$. For such points the Ordering Lemma dictates that either $a = 1$ or $p_A = \lambda$, and similarly that $c = 0$ or $p_C = \lambda$.

We separate our analysis of non-trivial equilibria into two parts. In the first subsection we determine equilibria that are admissible given the behavior of the system. This work will show that the only potential candidates are the steady-state solutions when $(p_A, p_B, p_C) = (\lambda, \lambda, \lambda)$ or when $(a, c) = (1, 0)$ and $p_B = \lambda$. In the second and third subsections we perform a linear stability analysis on these two possible solutions to determine their stability with respect to a , b , and c .

7.1 Existence of Non-Trivial Equilibria

As demonstrated previously, the condition that $(I_A, I_B, I_C) = (0, 0, 0)$ restricts the set of steady-state solutions to four possibilities:

$$\begin{array}{ll} \{p_A = \lambda, p_B = \lambda, c = 0\} & \{a = 1, p_B = \lambda, p_C = \lambda\} \\ \{p_A = \lambda, p_B = \lambda, p_C = \lambda\} & \{a = 1, p_B = \lambda, c = 0\}. \end{array}$$

We begin with the case when $\{p_A = \lambda, p_B = \lambda, c = 0\}$. We will show that these conditions do not satisfy the Ordering Lemma, and hence are unreachable by our

system. We begin by setting $p_A = \lambda$ and $p_B = \lambda$, given the fact that $c = 0$.

$$\begin{cases} \lambda = p_A &= \left(\frac{1+q-q^2}{1+q} \right) a + \left(\frac{q^2}{1+q} \right) b \\ \lambda = p_B &= \left(\frac{q^2+2q^3-q^4}{1+3q+2q^2} \right) a + \left(\frac{1+3q-4q^3+2q^4}{1+3q+2q^2} \right) b \end{cases}$$

We have then that the two right-hand sides are equal, and so we can solve for a in terms of b and q as

$$a = \left(\frac{1+3q-q^2-6q^3+2q^4}{1+3q-4q^3+q^4} \right) b. \quad (32)$$

The Ordering Lemma dictates that $a \geq b$, so either $a = b = 0$ or

$$\begin{aligned} \frac{1+3q-q^2-6q^3+2q^4}{1+3q-4q^3+q^4} &\geq 1 \\ 1+3q-q^2-6q^3+2q^4 &\geq 1+3q-4q^3+q^4 \\ q^2+2q^3-q^4 &\leq 0. \end{aligned}$$

However, since $q \in (0, 1)$, we have that

$$q^2+2q^3-q^4 > q^2+q^3 > 0.$$

Therefore equation (32) can only be true if $a = b = c = 0$, which is a trivial equilibrium.

Next we consider the case when $\{a = 1, p_B = \lambda, p_C = \lambda\}$. As before, we consider the system of equations when $p_C = \lambda$ and $p_B = \lambda$, given that $a = 1$.

$$\begin{cases} \lambda = p_C &= \left(\frac{1+q-q^2}{1+q} \right) c + \left(\frac{q^2}{1+q} \right) b \\ \lambda = p_B &= \left(\frac{q^2+2q^3-q^4}{1+3q+2q^2} \right) (c+1) + \left(\frac{1+3q-4q^3+2q^4}{1+3q+2q^2} \right) b \end{cases}$$

Again, we set the two right-hand sides equal to each other and solve for c in terms of b , leading to

$$c = \left(\frac{1+3q-q^2-6q^3+2q^4}{1+3q-4q^3+q^4} \right) b + \left(\frac{q^2+2q^3-q^4}{1+3q-4q^3+q^4} \right). \quad (33)$$

The Ordering Lemma tells us that $b \geq c$, so

$$b \geq \left(\frac{1+3q-q^2-6q^3+2q^4}{1+3q-4q^3+q^4} \right) b + \left(\frac{q^2+2q^3-q^4}{1+3q-4q^3+q^4} \right).$$

Solving for b yields the inequality

$$\left(\frac{q^2 + 2q^3 - q^4}{1 + 3q - 4q^3 + q^4} \right) b \geq \frac{q^2 + 2q^3 - q^4}{1 + 3q - 4q^3 + q^4}$$

$$b \geq 1.$$

Since b is a population fraction, $b \leq 1$, so this can only be true if $b = 1$. In this case, equation (33) tells us that

$$c = \left(\frac{1 + 3q - q^2 - 6q^3 + 2q^4}{1 + 3q - 4q^3 + q^4} \right) + \left(\frac{q^2 + 2q^3 - q^4}{1 + 3q - 4q^3 + q^4} \right) = 1.$$

Therefore $a = b = c = 1$, which is a trivial equilibrium.

We now consider the final two possibilities for equilibria. The case when $p_A = p_B = p_C = \lambda$ is equivalent to the steady-state solution $(a, b, c) = (\lambda, \lambda, \lambda)$. This can be seen most clearly when considering $p_A = p_C$, in which case

$$\left(\frac{1 + q - q^2}{1 + q} \right) a + \left(\frac{q^2}{1 + q} \right) b = \left(\frac{1 + q - q^2}{1 + q} \right) c + \left(\frac{q^2}{1 + q} \right) b$$

$$a = c.$$

The Ordering Lemma then tells us that $a \geq b \geq c$, and so it must be that $a = b = c = \lambda$ in order for $p_A = p_B = p_C = \lambda$. It is easy to show that for any particular $\varepsilon > 0$, this equilibrium is not reachable given the constraints of the Ordering Lemma. In order to show this fact, we consider Site A. Since $a_1 > \lambda$ under the initial conditions of the system, the TIT-FOR-TAT population in Site A can only reach λ if $I_A(a_n) < 0$ for some round n . This requires that $p_A < \lambda$, and so by the Ordering Lemma $p_C < p_B < p_A < \lambda$. Hence in the following round, all three TIT-FOR-TAT populations will decrease, and so p_A will remain less than λ . This argument can then be reapplied to show that all three populations will continue to decrease, hence $c_n < \lambda$ and only decreases in subsequent rounds.

Though this candidate for an equilibrium is unreachable given our particular initial conditions, we will still analyze its stability in the following subsection. We will also investigate the case when $\{a = 1, p_B = \lambda, c = 0\}$ as this potential equilibrium does not contradict the Ordering Lemma, hence it is an admissible steady-state solution.

7.2 Linear Stability Analysis of $\{a = 1, p_B = \lambda, c = 0\}$

Now that we have established that only two of the four non-trivial equilibria are admissible in the three-site scenario according to the Ordering Lemma, we investigate the linear stability of these equilibria. Let us define $\mathcal{I} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ as the function that updates population numbers in the three sites:

$$\mathcal{I}(a_n, b_n, c_n) = (a_{n+1}, b_{n+1}, c_{n+1}).$$

We'll denote the first component function of this mapping by $\mathcal{I}_A(a, b, c)$, and similarly for B and C. Using the form of the increment function introduced by equation (18) but suppressing its dependence on w , which is constant in any particular multi-site tournament, we have $\mathcal{I}_A = a + I_A(a, p_A)$. Note that \mathcal{I}_A is independent of c , and that it depends on b only through p_A . The second and third component functions of \mathcal{I} are $\mathcal{I}_B = b + I_B(b, p_B)$ and $\mathcal{I}_C = c + I_C(c, p_C)$, respectively. Using this notation, we write the Jacobian of \mathcal{I} as

$$\mathcal{J} = \begin{bmatrix} \frac{\partial \mathcal{I}_A}{\partial a} & \frac{\partial \mathcal{I}_A}{\partial b} & \frac{\partial \mathcal{I}_A}{\partial c} \\ \frac{\partial \mathcal{I}_B}{\partial a} & \frac{\partial \mathcal{I}_B}{\partial b} & \frac{\partial \mathcal{I}_B}{\partial c} \\ \frac{\partial \mathcal{I}_C}{\partial a} & \frac{\partial \mathcal{I}_C}{\partial b} & \frac{\partial \mathcal{I}_C}{\partial c} \end{bmatrix}.$$

We begin by considering the equilibrium at which $\{a = 1, p_B = \lambda, c = 0\}$. Let b^* represent the TIT-FOR-TAT population required in Site B so that $p_B = \lambda$. Given that $a = 1$ and $c = 0$, the equation $\lambda = p_B$ can be written as

$$\lambda = \frac{q^2 + 2q^3 - q^4}{1 + 3q + 2q^2} + \frac{1 + 3q - 4q^3 + 2q^4}{1 + 3q + 2q^2} b^*.$$

Solving for b^* ,

$$b^* = \frac{(1 + 3q + 2q^2)\lambda - (q^2 + 2q^3 - q^4)}{1 + 3q - 4q^3 + 2q^4}. \quad (34)$$

Now we use the Jacobian matrix to analyze the linear stability at $(1, b^*, 0)$. Since the lateral site increments are independent of the each other's TIT-FOR-TAT population,

we know that both $\partial \mathcal{I}_A / \partial c$ and $\partial \mathcal{I}_C / \partial a$ are 0. In this case, we also see that

$$\left. \frac{\partial \mathcal{I}_A}{\partial p_A} \right|_{a=1} = \frac{\partial}{\partial p_A} \left(\frac{a(1-a)(p_A - \lambda)}{p_A a + 2 + 4\lambda + \lambda(3p_A - a)} \right) \Big|_{a=1} = 0 \quad (35)$$

due to the factor of $(1-a)$ in the numerator. Therefore,

$$\frac{\partial \mathcal{I}_A}{\partial b} = \frac{\partial \mathcal{I}_A}{\partial b} + \frac{\partial \mathcal{I}_A}{\partial p_A} \frac{\partial p_A}{\partial b} = 0 + 0 \cdot \frac{\partial p_A}{\partial b} = 0.$$

A similar analysis shows that the partial derivative of \mathcal{I}_C with respect to b is 0, so the matrix \mathcal{J} has the form

$$\mathcal{J} = \begin{bmatrix} \frac{\partial \mathcal{I}_A}{\partial a} & 0 & 0 \\ \frac{\partial \mathcal{I}_B}{\partial a} & \frac{\partial \mathcal{I}_B}{\partial b} & \frac{\partial \mathcal{I}_B}{\partial c} \\ 0 & 0 & \frac{\partial \mathcal{I}_C}{\partial c} \end{bmatrix}.$$

The characteristic polynomial tells us that the eigenvalues of this matrix are precisely the diagonal elements, which we consider next. We established that $\partial \mathcal{I}_A / \partial p_A = 0$ in equation (35), so

$$\begin{aligned} \frac{\partial \mathcal{I}_A}{\partial a} &= 1 + \frac{\partial \mathcal{I}_A}{\partial a} + \frac{\partial \mathcal{I}_A}{\partial p_A} \frac{\partial p_A}{\partial a} \\ &= 1 + \frac{\partial}{\partial a} \left(\frac{a(1-a)(p_A - \lambda)}{p_A a + 2 + 4\lambda + \lambda(3p_A - a)} \right) + 0. \end{aligned} \quad (36)$$

We consider this derivative using Quotient Rule by letting $f = a(1-a)(p_A - \lambda)$ and $g = p_A a + 2 + 4\lambda + \lambda(3p_A - a)$. Since we are evaluating at $a = 1$, it follows that $f = 0$, and so the form of Quotient Rule becomes

$$\left(\frac{f}{g} \right)' \Big|_{a=1} = \frac{f'g - fg'}{g^2} \Big|_{a=1} = \frac{f'g}{g^2} \Big|_{a=1} = \frac{f'}{g} \Big|_{a=1},$$

where the prime notation indicates differentiation with respect to a . We calculate f' as

$$f' \Big|_{a=1} = (1-a)(p_A - \lambda) - a(p_A - \lambda) \Big|_{a=1} = -(p_A - \lambda).$$

When we substitute this into (36) we see that

$$\left. \frac{\partial \mathcal{I}_A}{\partial a} \right|_{a=1} = 1 + \frac{-(p_A - \lambda)}{p_A + 2 + 4\lambda + \lambda(3p_A - 1)} < 1$$

since $p_A > \lambda$. Similarly

$$\begin{aligned}\frac{\partial \mathcal{I}_C}{\partial c} &= 1 + \frac{\partial I_C}{\partial c} + \frac{\partial I_C}{\partial p_C} \frac{\partial p_C}{\partial c} \\ &= 1 + \frac{(1-c)(p_C - \lambda)}{4 + (1-w)(3p_C)} < 1\end{aligned}$$

since $p_C < \lambda$. Lastly, we consider

$$\frac{\partial \mathcal{I}_B}{\partial b} = 1 + \frac{\partial I_B}{\partial b} + \frac{\partial I_B}{\partial p_B} \frac{\partial p_B}{\partial b}.$$

Isolating the second term in this derivative, we see that

$$\begin{aligned}\left. \frac{\partial I_B}{\partial b} \right|_{p_B=\lambda} &= \left. \frac{\partial}{\partial b} \left(\frac{b(1-b)(p_B - \lambda)}{p_B b + 2 + 4\lambda + \lambda(3p_B - b)} \right) \right|_{p_B=\lambda} \\ &= \left[(p_B - \lambda) \left(\frac{b(1-b)}{p_B b + 2 + 4\lambda + \lambda(3p_B - b)} \right) \right] \Big|_{p_B=\lambda} \\ &= 0.\end{aligned}$$

This simplifies the form of $\frac{\partial \mathcal{I}_B}{\partial b}$ to

$$\frac{\partial \mathcal{I}_B}{\partial b} = 1 + \frac{\partial I_B}{\partial p_B} \frac{\partial p_B}{\partial b}.$$

We know from Lemma 2 that $\partial I_B / \partial p_B > 0$, and since p_B is simply a weighted average of a , b , and c , we know that the coefficient of the weighted average isolated by $\partial p_B / \partial b$ is positive as well. Hence

$$\frac{\partial \mathcal{I}_B}{\partial b} > 1.$$

With two eigenvalues less than 1, and one eigenvalue larger than 1, we see that this equilibrium is semi-stable. In particular, since

$$\begin{bmatrix} \frac{\partial \mathcal{I}_A}{\partial a} & 0 & 0 \\ \frac{\partial \mathcal{I}_B}{\partial a} & \frac{\partial \mathcal{I}_B}{\partial b} & \frac{\partial \mathcal{I}_B}{\partial c} \\ 0 & 0 & \frac{\partial \mathcal{I}_C}{\partial c} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\partial \mathcal{I}_B}{\partial b} \\ 0 \end{bmatrix},$$

the direction of instability is in the b -direction. However, since we are working with a discrete dynamical system, it is possible, in principle, to “jump” the region of instability

in the b -direction and then approach the equilibrium along a stable manifold. In the following theorem we establish that it is neither possible to step on to the equilibrium nor to approach it in the limit.

Theorem 3. *The point $(1, b^*, 0)$ is inaccessible in the three-site line.*

Proof. We begin by showing that it is not possible to step onto $(1, b^*, 0)$ even if the system approaches it in the limit. Suppose that $a_n \neq 1$. In order to step onto the equilibrium, there must be some round n such that

$$\begin{aligned} 1 - a_n &= I_A(a_n) \\ 1 - a_n &= \frac{a_n(1 - a_n)(p_A - \lambda)}{p_A a_n + 2 + 4\lambda + \lambda(3p_A - a_n)} \\ 1 &= \frac{a_n(p_A - \lambda)}{p_A a_n + 2 + 4\lambda + \lambda(3p_A - a_n)}. \end{aligned}$$

Solving for p_A , we see that this requires

$$p_A = \frac{-(2 + 4\lambda)}{3\lambda} < 0.$$

This is clearly impossible in our system, and so if $a_1 \neq 1$, then $a_n \neq 1$ for all $n \in \mathbb{N}$. A similar analysis demonstrates that $I_C(c_n) \neq -c_n$ if $c_1 \neq 0$, and so it is not possible to step onto the equilibrium.

We now consider the case where the system approaches $(1, b^*, 0)$ along some stable manifold. Suppose that for some round n , $b_n = b^*$. In order for $b_{n+1} = b^*$ also, it must be that $\lambda = p_B$, which we write as

$$\lambda = \left(\frac{q^2 + 2q^3 - q^4}{1 + 3q + 2q^2} \right) (a_n + c_n) + \frac{1 + 3q - 4q^3 + 2q^4}{1 + 3q + 2q^2} b^*. \quad (37)$$

After substituting in the value of b^* from equation (34) and simplifying, we see that equation (37) is equivalent to

$$1 = a_n + c_n.$$

Suppose that in round n , we have that $a_n + c_n = 1$ and $b_n = b^*$, so $p_B = \lambda$. Then in round $n + 1$, we still have that $b_{n+1} = b^*$, but in order for $a_{n+1} + c_{n+1} = a_n + I_A(a_n) +$

$c_n + I_C(c_n) = 1$, it must be that $I_A(a_n) + I_C(c_n) = f(a_n, c_n) = 0$. If $f(a_n, c_n) \neq 0$, then $a_{n+1} + c_{n+1} \neq 1$, and so $p_B \neq \lambda$. It follows that $b_{n+2} \neq b^*$.

Let $N \in \mathbb{N}$ and suppose that for all $n \geq N$, $b_n = b^*$. This can only be true if the point $(a_n, c_n) \neq (1, 0)$ lies on the line $a_n + c_n = 1$ and on the level curve $f(a_n, c_n) = 0$ for all $n \geq N$. Since this must be true for infinitely many n , there must be infinitely many intersections of the two curves. A direct application of the Mean Value Theorem tells us that this implies there must be infinitely many places where the slope of the tangent line to the level curve equals the slope of the line, -1 . So let us determine values of a for which

$$\frac{-\partial f/\partial c}{\partial f/\partial a} = -1 \longrightarrow \frac{\partial f}{\partial c} = \frac{\partial f}{\partial a} \quad (38)$$

given that $c = 1 - a$ and $b = b^*$. Recall that

$$\begin{aligned} \frac{\partial f}{\partial a} &= \frac{\partial I_A}{\partial a} + \frac{\partial I_C}{\partial a} \\ &= \frac{\partial}{\partial a} \left(\frac{a(1-a)(p_A - \lambda)}{p_A a + 2 + 4\lambda + \lambda(3p_A - a)} \right) \end{aligned}$$

Using the fact that $b = b^*$ and the definition of p_A from equation (13), we see that f is just a rational expression in both a and c , so its derivatives will be rational expressions as well. Substituting $c = 1 - a$ yields rational expressions in a for both $\partial f/\partial a$ and $\partial f/\partial c$, so we can multiply equation (38) by the common denominator of these fractions, and then collect all the terms on one side of the equation to write it as $\mathcal{P}(a) = 0$, where $\mathcal{P}(a)$ is some polynomial in a . The Fundamental Theorem of Algebra tells us that there are only finitely many values of a that satisfy this equation, and so there cannot be infinitely many intersections of the line $a + c = 1$ and the curve $f(a, c) = 0$. Therefore for any $N \in \mathbb{N}$, there exists an $k \geq N$ such that $b_k \neq b^*$.

This result tells us that it is not possible for b_n to remain b^* as $a_n \rightarrow 1$ and $c_n \rightarrow 0$, and so the middle site population will consistently jump off of b^* . Nor can these perturbations converge to 0 as $(a, c) \rightarrow (1, 0)$ as this would require the point (a_n, b_n, c_n) to enter the linear regime of the equilibrium, and the linear stability analysis shows that the equilibrium is linearly unstable in the b -direction. Therefore it is not possible

to converge to $\{a = 1, p_B = \lambda, c = 0\}$ from any direction. \square

7.3 Linear Stability Analysis of $\{a = \lambda, b = \lambda, c = \lambda\}$

As with the previous equilibrium, we begin by analyzing the Jacobian of the system given $a = b = c = \lambda$. In this case, the partial derivative of \mathcal{I}_A with respect to a is

$$\frac{\partial \mathcal{I}_A}{\partial a} = 1 + \frac{\partial I_A}{\partial a} + \frac{\partial I_A}{\partial p_A} \frac{\partial p_A}{\partial a}$$

where the constituent terms are

$$\left. \frac{\partial I_A}{\partial a} \right|_{p_A=\lambda} = (p_A - \lambda) \cdot \left. \frac{\partial}{\partial a} \left(\frac{a(1-a)}{p_A a + 2 + 4\lambda + \lambda(3p_A - a)} \right) \right|_{p_A=\lambda} = 0,$$

and

$$\begin{aligned} \left. \frac{\partial I_A}{\partial p_A} \right|_{p_A=\lambda} &= \left. \frac{a(1-a)}{p_A a + 2 + 4\lambda + \lambda(3p_A - a)} \right|_{p_A=\lambda, a=\lambda} \\ &= \frac{\lambda(1-\lambda)}{2 + 4\lambda + 3\lambda^2}, \end{aligned}$$

and

$$\frac{\partial p_A}{\partial a} = \frac{1 + q - q^2}{1 + q}.$$

Consequently, in the case when $a = b = c = \lambda$, the first element of the Jacobian matrix has the form

$$\mathcal{J}_{1,1} = 1 + \Lambda \mathcal{Q}_1^A$$

where

$$\Lambda = \frac{\lambda(1-\lambda)}{2 + 4\lambda + 3\lambda^2} \quad \text{and} \quad \mathcal{Q}_1^A = \frac{1 + q - q^2}{1 + q}.$$

We continue this process by considering the second entry in the Jacobian. Again we have that

$$\frac{\partial \mathcal{I}_A}{\partial b} = \frac{\partial I_A}{\partial b} + \frac{\partial I_A}{\partial p_A} \frac{\partial p_A}{\partial b},$$

where we consider the value of each partial derivative. In this case, since I_A does not directly depend on b , we have that $\partial I_A / \partial b = 0$. We also note that the term $\partial I_A / \partial p_A$ is

unchanged from the first derivative, and so we must only calculate the value of the third derivative as

$$\frac{\partial p_A}{\partial b} = \frac{q^2}{1+q}.$$

We now see that the second entry to the Jacobian has the form

$$\mathcal{J}_{1,2} = \Lambda \mathcal{Q}_2^A$$

where Λ is defined as before and

$$\mathcal{Q}_2^A = \frac{q^2}{1+q}.$$

Finally, since I_A is independent of c , the third entry of the Jacobian will be 0. The remaining elements of the Jacobian matrix are derived similarly, and have similar form. The value of Λ is the same for each element of the matrix since $\partial I / \partial p$ is site-independent and we are considering the case when $a = b = c = \lambda$. By contrast, the \mathcal{Q} -terms vary but exhibit symmetry between the lateral sites. In particular, the \mathcal{Q} -terms are simply the coefficients in the weighted averages p_A , p_B , and p_C . With this understanding, we write the Jacobian matrix as

$$\mathcal{J} = \begin{bmatrix} 1 + \Lambda \mathcal{Q}_1^A & \Lambda \mathcal{Q}_2^A & 0 \\ \Lambda \mathcal{Q}_1^B & 1 + \Lambda \mathcal{Q}_2^B & \Lambda \mathcal{Q}_1^B \\ 0 & \Lambda \mathcal{Q}_2^A & 1 + \Lambda \mathcal{Q}_1^A \end{bmatrix}$$

where

$$\begin{aligned} \mathcal{Q}_1^A &= \frac{1+q-q^2}{1+q} & \mathcal{Q}_2^A &= \frac{q^2}{1+q} \\ \mathcal{Q}_1^B &= \frac{q^2+2q^3-q^4}{1+3q+2q^2} & \mathcal{Q}_2^B &= \frac{1+3q-4q^3+2q^4}{1+3q+2q^2}. \end{aligned}$$

The structure of this matrix makes it easy to pick out possible eigenvectors for the system. In particular, we see that both $\langle 1, 0, -1 \rangle$ and $\langle 1, 1, 1 \rangle$ are eigenvectors since

$$\begin{bmatrix} 1 + \Lambda \mathcal{Q}_1^A & \Lambda \mathcal{Q}_2^A & 0 \\ \Lambda \mathcal{Q}_1^B & 1 + \Lambda \mathcal{Q}_2^B & \Lambda \mathcal{Q}_1^B \\ 0 & \Lambda \mathcal{Q}_2^A & 1 + \Lambda \mathcal{Q}_1^A \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 + \Lambda \mathcal{Q}_1^A \\ 0 \\ -1 - \Lambda \mathcal{Q}_1^A \end{bmatrix}$$

and

$$\begin{bmatrix} 1 + \Lambda \mathcal{Q}_1^A & \Lambda \mathcal{Q}_2^A & 0 \\ \Lambda \mathcal{Q}_1^B & 1 + \Lambda \mathcal{Q}_2^B & \Lambda \mathcal{Q}_1^B \\ 0 & \Lambda \mathcal{Q}_2^A & 1 + \Lambda \mathcal{Q}_1^A \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + \Lambda(\mathcal{Q}_1^A + \mathcal{Q}_2^A) \\ 1 + \Lambda(2\mathcal{Q}_1^B + \mathcal{Q}_2^B) \\ 1 + \Lambda(\mathcal{Q}_1^A + \mathcal{Q}_2^A) \end{bmatrix} = \begin{bmatrix} 1 + \Lambda \\ 1 + \Lambda \\ 1 + \Lambda \end{bmatrix}.$$

This also tells us that the eigenvalues corresponding to $\langle 1, 0, -1 \rangle$ and $\langle 1, 1, 1 \rangle$ are $1 + \Lambda \mathcal{Q}_1^A$ and $1 + \Lambda$, respectively. Both of these eigenvalues are greater than 1, so they indicate instability in the corresponding directions. The final eigenvector is not as obvious, and so we consider an eigenvector of the form $\langle 1, x, 1 \rangle$, where $x \in \mathbb{R}$. We determine the value of x by using the system

$$\begin{bmatrix} 1 + \Lambda \mathcal{Q}_1^A & \Lambda \mathcal{Q}_2^A & 0 \\ \Lambda \mathcal{Q}_1^B & 1 + \Lambda \mathcal{Q}_2^B & \Lambda \mathcal{Q}_1^B \\ 0 & \Lambda \mathcal{Q}_2^A & 1 + \Lambda \mathcal{Q}_1^A \end{bmatrix} \begin{bmatrix} 1 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + \Lambda(\mathcal{Q}_1^A + x\mathcal{Q}_2^A) \\ x + \Lambda(2\mathcal{Q}_1^B + x\mathcal{Q}_2^B) \\ 1 + \Lambda(\mathcal{Q}_1^A + x\mathcal{Q}_2^A) \end{bmatrix}. \quad (39)$$

Since the proposed eigenvector has the form $\langle 1, x, 1 \rangle$, the ratio between the second and first components on the right-hand side of equation (39) must be $x : 1$. That is

$$x(1 + \Lambda(\mathcal{Q}_1^A + x\mathcal{Q}_2^A)) = x + \Lambda(2\mathcal{Q}_1^B + x\mathcal{Q}_2^B).$$

This is a quadratic polynomial in x , namely

$$(\mathcal{Q}_2^A)x^2 + (\mathcal{Q}_1^A - \mathcal{Q}_2^B)x - 2\mathcal{Q}_1^B = 0. \quad (40)$$

We already have that one root of this polynomial is $x = 1$ since $\langle 1, 1, 1 \rangle$ is an eigenvector of the system. Thus we can factor out the quantity $(x - 1)$, leaving us with

$$(x - 1)(\mathcal{Q}_2^A x + 2\mathcal{Q}_1^B) = 0 \Rightarrow x = \frac{-2\mathcal{Q}_1^B}{\mathcal{Q}_2^A} = \frac{-2(1 + 2q - q^2)}{1 + 2q}.$$

Note that the linear term that results from expanding the product on the left-hand side of this equation is

$$2\mathcal{Q}_1^B - \mathcal{Q}_2^A = (1 - \mathcal{Q}_2^B) - (1 - \mathcal{Q}_1^A) = \mathcal{Q}_1^A - \mathcal{Q}_2^B,$$

which agrees with equation (40). Hence the second solution to equation (40) is

$$x = \frac{-2\mathcal{Q}_1^B}{\mathcal{Q}_2^A} = \frac{-2(1 + 2q - q^2)}{1 + 2q}.$$

The corresponding eigenvalue is just $1 + \Lambda(\mathcal{Q}_1^A + x\mathcal{Q}_2^A)$. It is possible that this value is less than 1 for particular x . In particular, consider the case when $q = 1$. Then

$$x = \frac{-2(2)}{3} = -\frac{4}{3}, \quad \mathcal{Q}_1^A = \frac{1}{2}, \quad \text{and} \quad \mathcal{Q}_2^A = \frac{1}{2},$$

and so $|\mathcal{Q}_1^A + x\mathcal{Q}_2^A| = |\frac{1}{2} - \frac{4}{3}(\frac{1}{2})| = |-\frac{1}{6}| < 1$. Since all of these values change continuously with q , it is clear that for some large values of q , this equilibrium will have a stable direction. However, the constraint of the Ordering Lemma limits the directions from which we can approach the equilibrium, and the stable eigenvector is tangential to the wedge representing valid 3-tuples for our system, which means that it is not possible to approach the equilibrium given our initial conditions.

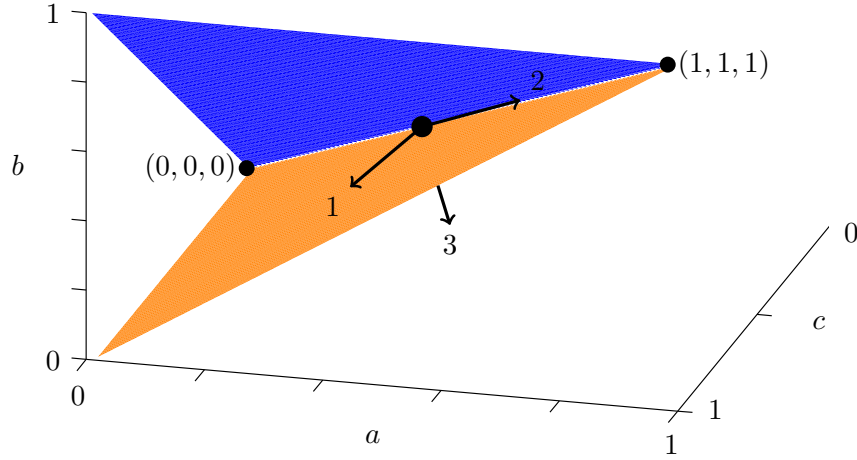


Figure 12: The region of permissible 3-tuples given the Ordering Lemma (blue & orange) and the three eigenvectors for the equilibrium at $(\lambda, \lambda, \lambda)$ numbered in the order they are discussed.

The fact that this stability is present seems to indicate that there may be other initial conditions, not of the form

$$a_1 = \lambda + \varepsilon(1 - \lambda)$$

$$b_1 = \lambda$$

$$c_1 = \lambda - \varepsilon\lambda,$$

from which it is possible to approach a nontrivial equilibrium in Site B. In particular, this eigenvector implies that the stable equilibrium appears if q is sufficiently large and if $a, c > \lambda > b$, or vice versa. Consider the case when $a, c > \lambda > b$. In terms of the IPD tournament, this stable equilibrium corresponds to the situation when the two lateral sites are both skewed in favor of TIT-FOR-TAT players. In this case, the lateral site influence the player pool in the middle site, and so $p_B > \lambda$. However, the traveling population from Site B exerts a similarly significant influence on both of the individual sites, meaning that $p_A, p_C < \lambda$. Hence the fraction of TIT-FOR-TAT players in the lateral sites is reduced, but the fraction in the middle site rises. This draws the three-site line to the steady-state solution $(\lambda, \lambda, \lambda)$.

8 Numerical Estimates

The proof that $b_n \rightarrow 1$ when λ is sufficiently small seen in Section 5 requires that $0 < \lambda < \min\{\lambda_1, \lambda_2\}$, where λ_1 and λ_2 are the numbers guaranteed by Lemmas 4 and 5. Similarly, in Section 6 we establish that $b_n \rightarrow 0$ when $1 > \lambda > \max\{1/2, \lambda_3\}$, where λ_3 is guaranteed by Lemma 5b. In between “sufficiently small” and “sufficiently large” λ , b_n exhibits transient behavior as seen in Figure 8. In this section we investigate numerically which values of λ correspond to TIT-FOR-TAT dominance in the middle site. We also consider the effect of q on the behavior of the middle site population.

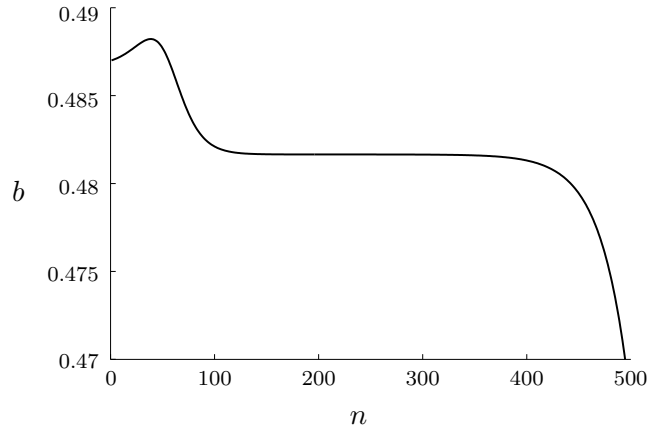


Figure 13: Graph of the transient behavior seen in the demographic of Site B for $\lambda = 0.487004566$.

Toward establishing this numerical evidence, we constructed a simulation environment in which we implemented the model proposed in this paper. These simulations were run in MATLAB[®] and used the process described by equation (8) in order to update each of the three populations. As described below, the limiting behavior of b_n was detected by observing its value cross a threshold.

We know from the Ordering Lemma that $a_n \geq b_n \geq c_n$, so for any value of b_n the

weighted average p_B is minimized when $a_n = b_n$ and $c_n = 0$. In this case,

$$\begin{aligned} p_B &= \mathcal{Q}_1^B(b_n) + \mathcal{Q}_2^B(b_n) + \mathcal{Q}_1^B(0) \\ &= (\mathcal{Q}_1^B + \mathcal{Q}_2^B)b_n. \end{aligned}$$

It follows that $p_B > \lambda$ when

$$b_n > \frac{\lambda}{\mathcal{Q}_1^B + \mathcal{Q}_2^B}.$$

Consequently when b_n is greater than this value, we have that $I_B > 0$ and so $b_{n+1} > b_n$. Therefore once b_n crosses this value, it will experience monotonic increase for all remaining rounds. Similarly, the weighted average p_B is maximized when $a_n = 1$ and $c_n = b_n$, in which case we have

$$\begin{aligned} p_B &= \mathcal{Q}_1^B(1) + \mathcal{Q}_2^B(b_n) + \mathcal{Q}_1^B(b_n) \\ &= \mathcal{Q}_1^B + (\mathcal{Q}_1^B + \mathcal{Q}_2^B)b_n. \end{aligned}$$

So $p_B < \lambda$ when

$$b_n < \frac{\lambda - \mathcal{Q}_1^B}{\mathcal{Q}_1^B + \mathcal{Q}_2^B}.$$

Consequently, the increment in b is $I_B < 0$, and so $b_{n+1} < b_n$. This brings about monotonic decrease in b_n for all remaining rounds.

The MATLAB[®] simulations were run for each (q, λ) pair in a 1000×1000 mesh of $[0, 1] \times [0, 1]$ in the $q\lambda$ -plane. Each simulation terminated when the value of b_n crossed one of the thresholds or the number of rounds exceeded 1200, and each (q, λ) pair was assigned two values: a 1 or 0 according to which threshold was crossed, and the round number n at which the simulation terminated.

The boundary between the two regions in Figure 14 represents the qualitative change in behavior across the $q\lambda$ -plane. When we interpolate these boundary points, we find that

$$\lambda(q) \approx 0.008634q^2 + 0.000750q + 0.484524.$$

This gives us that the critical value of λ where the behavior of the middle site changes depends on q and lies roughly in the interval $[0.484, 0.494]$. Further, this tells us that

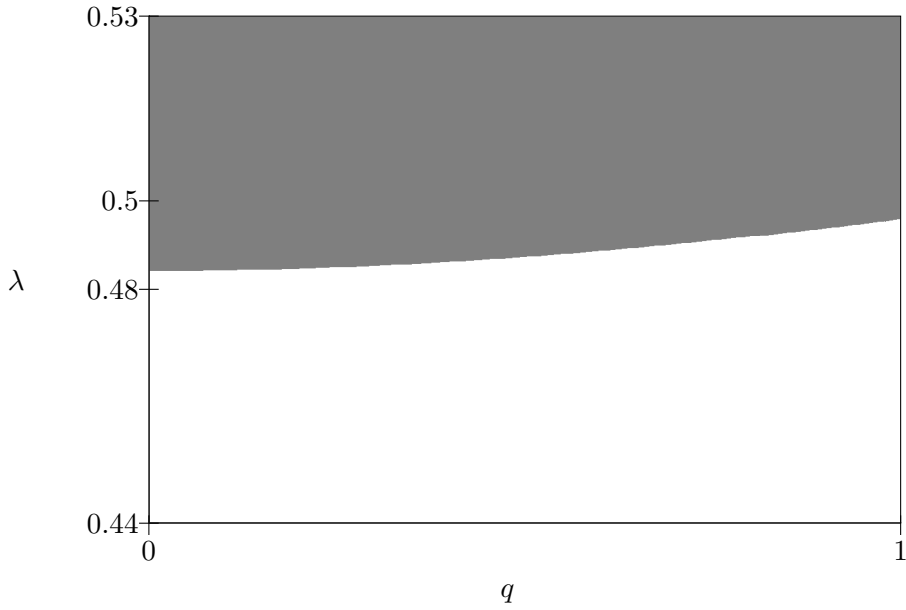


Figure 14: A colormap of the $q\lambda$ -plane where gray indicates that $b_n \rightarrow 0$ and white indicates $b_n \rightarrow 1$.

even though the critical λ value depends on q , it varies by approximately 2% as q ranges over $[0, 1]$.

Figure 15 shows that there is a ridge through the $q\lambda$ -plane that coincides with the boundary between the two regions in Figure 14. In particular, the contour plot in Figure 15 demonstrates that as the (q, λ) pairs approach the ridge, the number of rounds required for termination steadily increases. This indicates that even though q does not play a significant role in determining the long-term behavior of the middle site ($b \rightarrow 1$ or $b \rightarrow 0$), it affects the rate at which the demographics of the middle-site population change.

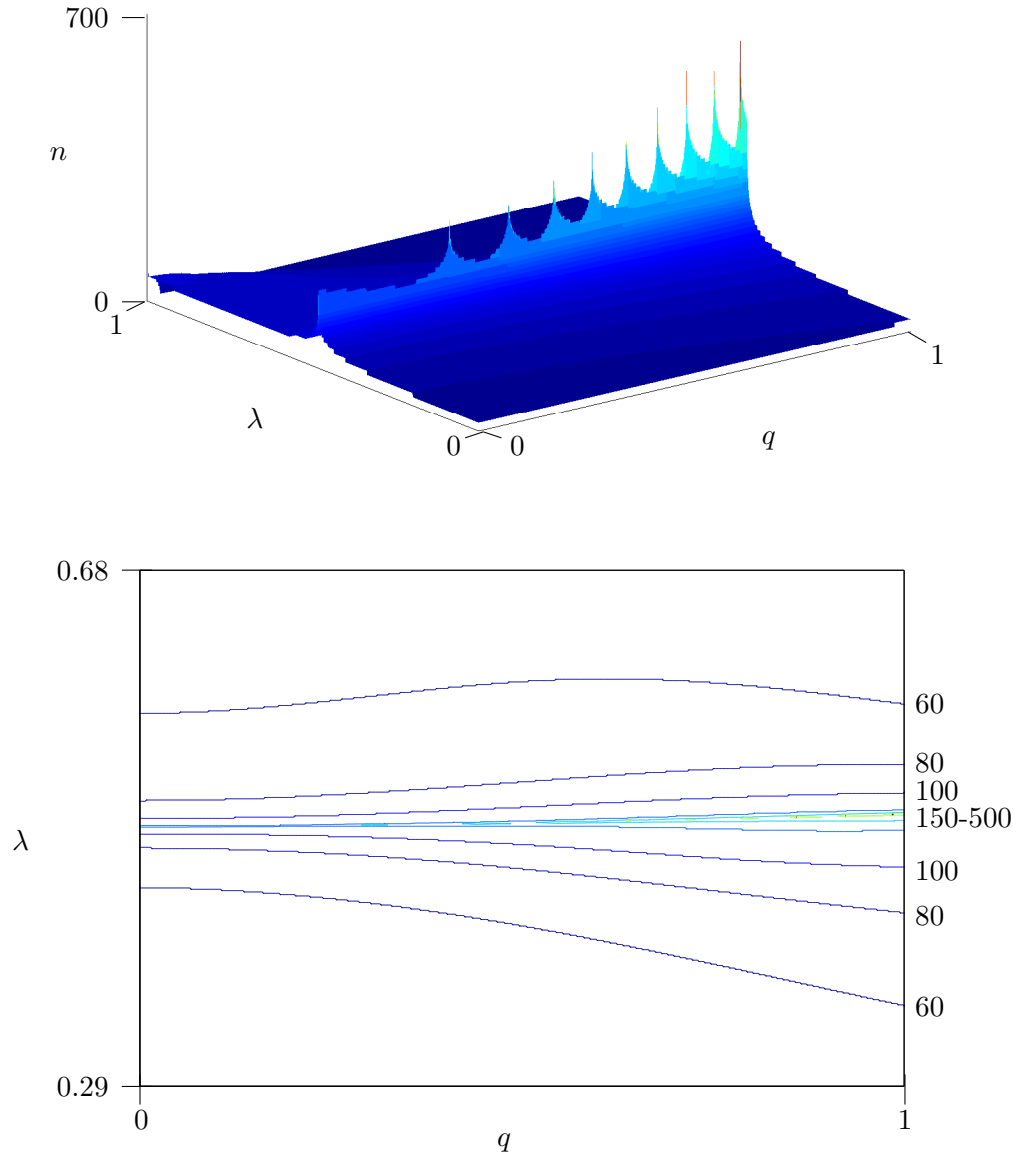


Figure 15: A surface plot (top) and corresponding contour plot (bottom) for the number of rounds (listed on the right) required for the IPD simulation to terminate.

9 Conclusion

Having ultimately determined the long-term behavior of the TIT-FOR-TAT population in Site B, we wish to translate these results back into the context of the IPD tournament. We have shown that for small λ , the TIT-FOR-TAT population in the middle site thrives and eventually pushes the MEAN population to extinction. On the other hand, when λ is large, the reverse scenario occurs and TIT-FOR-TAT players are driven to extinction. Having seen that there are no stable nontrivial equilibria in the system, we know that for any value of λ one population will eventually dominate the middle site, though this process may take hundreds or thousands of generations.

One way to interpret this behavior is to recall from equation 5 that λ and w have an inverse relationship, so we see that small values of λ correspond to large values of w , or in more direct terms, they correspond to longer games between players. This means that players in the tournament tend to have longer “relationships,” and so players that cooperate with each other are rewarded for that cooperation over time. TIT-FOR-TAT players form highly rewarding relationships with other TIT-FOR-TAT players in the community, and since these relationships tend to be lengthy, the benefits TIT-FOR-TAT players receive far outweigh the loss they experience when they encounter MEAN players.

Another possible interpretation is to note that small λ correspond to a large growth potential for the TIT-FOR-TAT population given the initial conditions. When λ is small, we see that the TIT-FOR-TAT population in Site A has significantly more room for growth than the MEAN population in Site C, and so over time Site A will begin to dominate the effective population for Site B.

The instability of the nontrivial equilibrium represents a strong limitation on the present model seeing as how real-world communities are in fact stable despite their diverse demographics. This limitation arises from the simplicity of the single-site game and the growth model. As soon as one of the population types gains an advantage

in the single-site tournament, that strategy will grow in size and will see even greater success in subsequent rounds. For this reason, the equilibrium at λ is unstable due to the structure of IPD, and this instability is preserved in some sense when the added complexity of interaction among multiple sites is introduced.

The work presented here has established a core result for understanding the behavior of multi-site IPD, but there are many elements of even this simple system that have yet to be explored. Possible future directions include exploring the effect of the payoff matrix on the behavior of the system, allowing each site to have different values of w or q , constructing new initial conditions that do lead to cooperative equilibria, including additional IPD strategies to the pool of player types, and adding stochastic elements to the model. Another possibility is to consider other geometries for arranging the sites, such as a ring or lattice. Understanding these more complex geometries would help in extending the results given here to more general setups of the multi-site IPD.

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